INTRODUCTION TO THE THEORY OF EXTENDED DYNAMICAL SYSTEMS

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Abstract

We consider nonlinear dissipative partial differential evolutions in unbounded domains, and in particular the solutions which stay bounded but do not necessarily tend to zero at infinity. The theory is developed by analogy with the case of finite dimensional dynamical systems.

1 Introduction.

Time evolution is a basic fact of every-days life. We observe all the time that most things (including ourselves) change with time. The laws governing these time evolutions at least in physical contexts have been established long time ago (like for example Newton's law). However they lead to evolution equations which have sometimes complicated (chaotic) solutions. The twentieth century has witnessed a large development of the Mathematical and Physical study of these complicated behaviours. A large amount of work in the Mathematics literature has been devoted to the study of "small" systems or more precisely systems where a small number of degrees of freedom are excited. For example systems with a small number of interacting particles like the famous three body problem of celestial mechanics.

There is however another important class of systems that can be called large (extended) systems which not only have many degrees of freedom excited but which have another built-in important property: the space dependence. One may think at first sight that these systems having many degrees of freedom excited are just much more complicated than the smaller ones in terms of their dynamical behaviour. This is of course true and perhaps the most striking illustration is fully developed turbulence. However it was soon observed in the early experiments that the complicated chaotic behaviours are sometimes superposed on more regular patterns called structures. These structures have peculiar properties which were observed experimentally and explained early after by the physicists. To describe these properties, I will use a simple example of these structures which is known to everyone although from a theoretical (and precise experimental) point of view it is one of the most difficult to study. This is the phenomenon of waves excited by wind on the sea or on large lakes. In the absence of wind or at very low wind velocity, there are no waves. This is not quite true as everybody knows but the small waves that are observed in this situation are due to other mechanisms. If the wind velocity grows, there is a threshold above which waves start to be excited by the wind. Precise observations show that the critical wind velocity (the velocity above which waves are excited) is always the same (reproducible). Moreover, the wavelength of the excited waves just above threshold is a constant, and is reproduced almost exactly in all observations. As mentioned before this has to be taken with a grain of salt because the observation is extremely difficult to perform with any satisfactory accuracy, but at least qualitatively everyone can observe these facts during vacations on the beach. This is called the Kelvin-Helmoltz instability. In the laboratory, a large variety of experiments on different systems reproduce this kind of behaviour and can be more precisely analysed. There are basically two important facts: first that there is a well defined threshold in a control parameter, and second that the typical size of the produced structures is also a fixed number (at least near threshold). This also discriminates a posteriori between small and large systems from the spatial point of view. In small systems, the size of the structures would be of the same order as the size of the spatial domain (container). Large systems are characterised by a size of the domain much larger than the typical size of the structure: wavelength on the sea are rarely larger than ten meters while the size of the domain is of the order of several hundreds or thousands of kilometres. One immediately notices that for systems in small spatial domains in the previous sense, boundary conditions will play an important role and in particular distort the structures if they appear at all. In large domains on the contrary one may hope that in a first approach, boundary conditions can be forgotten if one observes far away from the boundary, and it would be interesting to investigate the behaviour of the domain in an infinite system. This is reminiscent of the notion of thermodynamic limit in Statistical Mechanics, where some notions like phase transitions become much clearer, because of the very different relative sizes of the atomic and macroscopic (human) scales. These instability phenomena, and in particular the sharpness of the critical parameter are also very reminiscent of the bifurcation phenomenon occurring in low dimensional dynamical systems. This leads naturally to study the linear stability of the homogeneous state (the state before the structures show up). This has been done for a long time in particular by physicists and has lead to good predictions for the critical parameter values and for the wavelength at threshold. We refer for example to [5] [15] and [14] for many examples and references. In most of these studies, it was tacitly assumed that the system was of infinite spatial extent. In particular from a technical point of view this is manifest in the use of Fourier integrals instead of Fourier series or similar countable basis decomposition. More generally, one can try to study these infinitely extended systems as dynamical systems and see how much of the results for low dimensional systems can be adapted to this context.

It is therefore natural to try to define the evolution of infinite systems driven by non linear parabolic or hyperbolic evolution equations. Most of the results in the literature deal with bounded space domains or in the case of unbounded domains consider solutions in functions spaces which imply in some sense a decay at infinity, for example square integrable functions. However if we want for example to treat general waves, the natural space for the solutions will be a space of bounded functions (with some regularity). Our first task (section II) will therefore be to rigorously define the time evolution in unbounded domains and to prove the regularity of solutions.

When the time evolution is well defined, one may wonder if one can continue to study such systems following the lines of notions and results from the low dimensional dynamical systems. In other words, can one consider the partial differential equation as a dynamical system in an infinite dimensional phase space. A notion of (minimal) globally attracting set has been introduced to describe the asymptotic of solutions when viewed in a finite window. This is a set of functions (depending only on the space variables) which is globally invariant by the dynamics although each separate function may not be stationary. We will explain in section III how this (minimal) globally attracting set is constructed and show some simple examples of functions in it.

By analogy with the low dimensional systems, we can guess that varying a parameter in the system should lead to interesting phenomena similar to the bifurcations. We will discuss this phenomena of instabilities in section IV, first from a somewhat informal physical point of view, then in a rigorous way using a renormalisation group approach to derive the analog of the normal forms called the amplitude equation.

There are several quantities measuring the quantitative chaotic properties of low dimensional dynamical systems. For example one can look at the dimension of the attractor, the Lyapunov exponents which describe the growth rate of the distance between nearby initial conditions, the topological entropy which measures the variety of topologically different trajectories, or the metric entropy which measure the same quantity but restricted to typical trajectories of an ergodic invariant measure. When one tries to apply the standard definitions to infinite dimensional situations, one often finds an infinite result. A rough intuition is that in an extended system with structures of finite size, there is more or less one degree of freedom per structure and hence one gets infinitely many degrees of freedom. As in Statistical Mechanics, it is therefore useful to look for densities of the interesting quantities, or in other words, to prove that in some sense they have extensivity properties. It turns out that such ideas had been developed before to analyse function spaces by Kolmogorov and his collaborators (see [25]). This has lead for example to the notion of ϵ-entropy per unit volume, a notion already present in the fundamental paper by Shannon [30] on the entropy. We will show in section V how these ideas can be adapted to the study of attractors and dynamics of extended systems.

Many other results are by now available for systems defined in unbounded domains. For example there are many works devoted to the study of particular classes of interesting solutions like fronts, waves etc. We refer the reader to the literature for more details.

2 Existence and properties of the semi-flow.

In the Mathematical literature one finds a lot of papers devoted to the study of existence, uniqueness and properties of non linear partial differential equations in bounded space domain, where the data are given by the initial values and the boundary conditions. There are few results for systems in infinite domain. Some of them use classical function spaces which in some sense impose that the solutions tend to zero at infinity. This is not satisfactory if one wants to discuss waves for example. It would be more satisfactory to have results in spaces of bounded functions (with bounded derivatives for the regularity). In other words, we will have to consider the following questions and difficulties.

1. Prove existence, uniqueness and regularity of solutions in spaces of bounded functions (with bounded derivatives) in the infinite domain (think of the function space as L^∞).

2. The linear operators (linearised operators) that will appear are in general not compact and have continuous spectrum.

3. Attractors are infinite dimensional and we will need to reconsider the definition.

These questions have been investigated recently and we will discuss these problems one after the other without looking for the best results available at the moment (for which references will be provided) but concentrating on simple examples where one can understand more simply how the methods work.

2.1 Existence of the evolution semi-flow.

We start with the first problem of existence of the semi-flow of time evolution, namely given the initial condition, can one solve the evolution equation for positive times. Note that since there is no boundary (in some sense it is at infinity) there is no boundary condition. There are basically two techniques to deal with this question: the maximum principle and the method of local a-priori estimates (sometimes called the method of local energy estimates). The first method is rather elegant but restricted to particular classes of equations. I will not say more about it but refer to [27] for more information.

A-priori estimates are used very often in the case of bounded domains and therefore it is natural to try to use it also in this new context.

To explain the method, I will use an equation which is met very frequently (we will see why later on) and which is called the complex Ginzburg-Landau equation (cGL for short). This is an equation for a complex function $A(t,x)$ with time $t \in \mathbf{R}^+$ and space variable $x \in \mathbf{R}^d$. The equation depends on two fixed real parameters α and β and is given by

$$\partial_t A = (1+i\alpha)\Delta A + A - (1+i\beta)A|A|^2 \ . \tag{1}$$

We wish to prove that if we start with an initial condition A_0 which is a bounded function of x belonging moreover to a suitable function space (of functions not necessarily tending to zero at infinity), then we get a bounded solution for all positive times.

The first step is of course to prove a local time existence theorem. Let $C^0_{b,u}$ be the set of bounded uniformly continuous functions on $\mathbf{R}^d$ (similarly we will denote by $C^k_{b,u}$ the set of functions with k ($\in \mathbf{N}$) bounded and uniformly continuous derivatives).

Theorem 2.1. *Let $A_0 \in C^0_{b,u}$. Then there exists $T = T(A_0) > 0$ such that there exists a solution A of (1) on $[0,T] \times \mathbf{R}^d$, such that*

1. *$A(t,\,\cdot\,) \in C^2_{b,u}$ for any $T \in]0,T]$,*
2. $$\sup_{[0,T]} \big\|A(t,\,\cdot\,)\big\|_{L^\infty} \le 2\big\|A(0,\,\cdot\,)\big\|_{L^\infty}$$
3. *$A(t,\,\cdot\,)$ tends uniformly to $A(0,\,\cdot\,)$ when $t > 0$ decreases to zero.*

The proof uses a standard contraction mapping argument applied to the integral form of the equation and is left to the reader.

As we see in (*2*), the bound we obtain on the L^∞ norm deteriorates with time.

It is the role of the a-priori estimate to show that this is not the case. To try to imitate the bounded case, we will use a cut-off function which somehow localises the estimate. To simplify the notations, we will only derive the estimate for $d = 1$ There are many possibilities for the cut-off function, a convenient one is the function

$$\varphi(x) = \frac{b}{\pi(1+b^2x^2)} \ , \tag{2}$$

where b is a positive number to be suitably chosen later on. We now consider the function of time

$$L(t) = \int \varphi(x)|A(t,x)|^2 dx \ .$$

Note that from our choice of φ, this integral is convergent if $A(t,\cdot)$ is in L^∞. The idea is now to use this function as a sort of Lyapunov function $L(t)$, namely to try to say something about its time evolution. We have

$$\frac{d}{dt}L(t) = 2\Re\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx \ ,$$

and we would like to estimate the right hand side in terms of L. To do so, we use the evolution equation (1). We get

$$\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx$$

$$= (1+i\alpha)\int \varphi A'' A^* dx + \int \varphi|A|^2 dx - (1+i\beta)\int \varphi|A|^4 dx$$

$$= -(1+i\alpha)\int \varphi|A'|^2 - (1+i\alpha)\int \varphi' A^* A' dx + \int \varphi|A|^2 dx - (1+i\beta)\int \varphi|A|^4 dx \ .$$

Therefore

$$\Re\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx =$$

$$-\int \varphi|A'|^2 - \Re\left((1+i\alpha)\int \varphi' A^* A' dx\right) + \int \varphi|A|^2 dx - (1+i\beta)\int \varphi|A|^4 dx \ ,$$

$$\le -\int \varphi|A'|^2 + \sqrt{1+\alpha^2}\int |\varphi'||A^*||A'|dx + \int \varphi|A|^2 dx - \int \varphi|A|^4 dx \ .$$

We now observe that for any real Z we have (the proof is left to the reader)

$$-\frac{\varphi Z^2}{2} + \sqrt{1+\alpha^2}|\varphi'|Z|A^*| \le \frac{(1+\alpha^2)|\varphi'|^2}{2\varphi}|A|^2 \ .$$

Applying this inequality with $Z = A'(x)$ we get

$$\Re\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx$$

$$\leq \frac{1+\alpha^2}{2}\int \frac{|\varphi'|^2}{\varphi}|A|^2dx + \int \varphi|A|^2dx - \int \varphi|A|^4dx - \frac{1}{2}\int \varphi|A'|^2 \ .$$

It is at this point that the properties of the cut off function φ become important. With our particular choice, we have

$$\sup_x \frac{|\varphi'(x)|^2}{\varphi(x)^2} \leq b^2 \ .$$

Therefore

$$\Re\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx$$

$$\leq \left(1+\frac{(1+\alpha^2)b^2}{2}\right)\int \varphi|A|^2dx - \int \varphi|A|^4dx - \frac{1}{2}\int \varphi|A'|^2 \ .$$

The reader is again invited to check the simple estimate

$$\left(1+\frac{(1+\alpha^2)b^2}{2}\right)|A|^2 - \frac{|A|^4}{2} \leq \frac{1}{2}\left(2+\frac{(1+\alpha^2)b^2}{2}\right)^2 - |A|^2 \ ,$$

which implies immediately

$$\Re\int \varphi(x)A^*(t,x)\partial_t A(t,x)dx$$

$$\leq -\int \varphi|A|^2dx + \frac{1}{2}\left(2+\frac{(1+\alpha^2)b^2}{4}\right)^2\int \varphi dx - \int \varphi|A|^4dx - \frac{1}{2}\int \varphi|A'|^2 \ .$$

In other words, if we define the constant C by

$$C = \frac{1}{4}\left(2+\frac{(1+\alpha^2)b^2}{2}\right)^2\int \varphi dx \ ,$$

we obtain

$$\frac{dL}{dt} \leq -2L + 2C - \int \varphi|A'|^2 - \int \varphi|A|^4 \leq -2L + 2C \ . \tag{3}$$

Assume we know (see Theorem 2.1) that the solution exists on a time interval $[0,t_1]$. Then we conclude that for any $t\in[0,t_1]$, we have

$$L(t) \leq 2C\left(1-e^{-2t}\right) + e^{-2t}L(0) \leq 2C\left(1-e^{-2t}\right) + e^{-2t}\|A(0,\cdot)\|_{L^\infty}^2 \ ,$$

where the last estimate follows from the fact that the integral of φ is one. In particular, we see that if $\|A(0,\cdot)\|_{L^\infty}^2 > 2C$, this estimate gets better with time. However this is not enough to apply again Theorem 2.1 at time t_1 since we do not have yet an estimate for $\|A(0,\cdot)\|_{L^\infty}$. Fortunately the above argument can be improved. First, instead of the function φ we can use for any real a the function

$$\varphi_a(x) = \varphi(x-a) \ .$$

We can repeat the argument and observe that the bound does not depend on a. Therefore we get

$$\tilde{L}(t) = \sup_a \int \varphi_a(x)|A(t,x)|^2 dx \le 2C\left(1 - e^{-2t}\right) + e^{-2t}\|A(0,\,\cdot\,)\|_{L^\infty}^2 \ . \tag{4}$$

This is not yet enough to prove that the solution is bounded. However we can get more from the previous argument. From equation (3) we have for any interval I

$$\int_I |A'(t,x)|^2 dx \le \frac{1}{\sup_{x\in I}\varphi(x)}\left(-\frac{dL}{dt} + 2C\right) \ .$$

If we know the existence of the solution in a time interval $[t_0, t]$, we can integrate this inequality, and dropping negative terms on the right hand side we get

$$\int_{t_0}^t ds \int_I |A'(s,x)|^2 dx \le \frac{1}{\sup_{x\in I}\varphi(x)}\left(L(t_0) + 2C(t - t_0)\right)$$

We now recall a Sobolev inequality for the interval I. We have

$$\|A\|_{L^\infty(I)}^2 \le \frac{1}{|I|}\|A\|_{L^2(I)}^2 + 2\|A\|_{L^2(I)}\|A'\|_{L^2(I)}$$

$$\le \frac{1}{|I|\sup_{x\in I}\varphi(x)}L(t) + \frac{2}{\sup_{x\in I}\varphi(x)}L(t)^{1/2}\left(\int \varphi(x)|A'(t,x)|^2 dx\right)^{1/2} \ .$$

Exercise 2.1. Give a proof of the first inequality in two lines (hint start by integrating on the interval $[x_0, x]$ the derivative of A^2).

Combining the above estimates we get for a unit interval I

$$\int_{t_0}^t \|A\|_{L^\infty(I)}^4 ds$$

$$\le \left(\frac{1}{\sup_{x\in I}\varphi(x)}\right)^2 \int_{t_0}^t \left(2L(s)^2 + 8L(s)L(t_0) + 8CL(s)(s - t_0)\right) ds \ . \tag{5}$$

The main point here is that the right hand side contains only the function L for which we already have an estimate. On the other hand, the upper bound will get large if the interval $[t_0, t]$ becomes too large.

We will now use the function of space and (positive) time defined by

$$g_t(x) = \frac{1}{\sqrt{\pi(1 + i\alpha)t}} e^{-x^2/((1+i\alpha)t)} \ ,$$

with the standard definition of the complex square root (cut on the negative axis).

It is convenient for later reference to state the following simple lemma.

Lemma 2.1. *The function g_t has the following properties.*

1. *There is a constant $C_1 > 0$ such that for any $t > 0$*

$$\big|g_t(x)\big| \le C_1 \frac{1}{\sqrt{\pi(1+\alpha^2)^{1/2}t}} e^{-x^2/((1+\alpha^2)^{1/2}t)} ,$$

$$\sup_{t>0} \int \big|g_t(x)\big|\, dx \le C_1 ,$$

and

$$\int \big|g_t'(x)\big|\, dx \le \frac{C_1}{\sqrt{t}} .$$

2. *If the function $A(t,x)$ satisfies the equation $\partial_t A = (1+i\alpha)A'' + A$, and $A(t,\,\cdot\,)$ tends to $A_0(\,\cdot\,)$ in $C^0_{b,u}(\mathbf{R})$ when $t \searrow 0$, then for $t > 0$*

$$A(t,x) = e^t g_t * A_0(x) .$$

3. *If the function $A(t,x)$ satisfies the equation $\partial_t A = A'' + A + f(t,\,\cdot\,)$ where $f \in L^\infty$ and $A(t,\,\cdot\,)$ tends to $A_0(\,\cdot\,)$ in $C^0_{b,u}(\mathbf{R})$ when $t \searrow 0$, then for $t > 0$*

$$A(t,x) = e^t g_t * A_0(x) + \int_0^t e^{t-s} g_{t-s} * f(s,\,\cdot\,)(x)\, ds .$$

Exercise 2.2. Prove the Lemma. Of course *2*) is a direct consequence of 3), but it is wiser first to prove directly *3*).

Since the cGL equation is homogeneous in time, we can apply this lemma on any interval $[t_0, t]$ on which we already know that the solution exists. We conclude that

$$|A(t,0)| \le e^{t-t_0} \big|\big(g_{t-t_0} * A(t_0,\,\cdot\,)\big)(0)\big| \\ + |1+i\beta| \left| \int_{t_0}^t e^{t-s} \big(g_{t-s} * \big(A(s,\,\cdot\,)|A(s,\,\cdot\,)|^2\big)(0) ds \right| . \tag{6}$$

We will estimate separately the two terms on the right hand side.

Using 1) of Lemma 2.1 we have the trivial estimate

$$\big|e^{t-t_0}\big(g_{t-t_0} * A(t_0,\,\cdot\,)\big)(0)\big| \le C_1 e^{t-t_0} \|A(t_0,\,\cdot\,)\|_{L^\infty} .$$

This estimate is useful for small time intervals $[t_0, t]$ but on the other hand it still depends on an L^∞ norm on the right hand side. We can however do better due to the inequality valid for any x and $s > 0$

$$|g_s| \le \frac{1}{\sqrt{\pi(1+\alpha^2)^{1/2}s}} e^{-x^2/((1+\alpha^2)s)} \le C_2 \left(\frac{1}{\sqrt{s}} + \sqrt{s}\right) \varphi(x)$$

where C_2 is a positive constant which depends on b and α.

Exercise 2.3. Prove this inequality.

We now obtain

$$\begin{aligned}\left|e^{t-t_0}\big(g_{t-t_0} * A(t_0,\,\cdot\,)\big)(0)\right| &\le e^{t-t_0} C_2 \left(\frac{1}{\sqrt{t-t_0}} + \sqrt{t-t_0}\right) \left|\big(\varphi{*}A(t_0,\,\cdot\,)\big)(0)\right| \\ &\le e^{t-t_0} C_2 \left(\frac{1}{\sqrt{t-t_0}} + \sqrt{t-t_0}\right) L(t_0)^{1/2}\end{aligned}$$

where we have used the Schwarz inequality to obtain the last bound. By using as before a translate of the cut-off function φ we can obtain an estimate for any x and we conclude that

$$\begin{aligned}&\left\|e^{t-t_0} g_{t-t_0} * A(t_0,\,\cdot\,)\right\|_{L^\infty} \\ &\le \min\left\{C_1 e^{t-t_0}\|A(t_0,\,\cdot\,)\|_{L^\infty}\,,\ e^{t-t_0} C_2 \left(\frac{1}{\sqrt{t-t_0}} + \sqrt{t-t_0}\right) L(t_0)\right\}\ . \qquad (7)\end{aligned}$$

We now come to the estimate of the second term in (6). As we will see later on it is sufficient to restrict oneself to the case $0 < t - t_0 \le 1$. We will therefore estimate this term only under this hypothesis. We can decompose the convolution in an infinite sum as follows

$$\begin{aligned}&\int_{t_0}^{t} e^{t-s}\Big(g_{t-s} * \big(A(s,\,\cdot\,)|A(s,\,\cdot\,)|^2\big)\Big)(0)ds \\ &= \sum_{n\in\mathbf{Z}} \int_{t_0}^{t} e^{t-s} ds \int_{n-1/2}^{n+1/2} g_{t-s}(-y) A(s,y)|A(s,y)|^2 dy\ .\end{aligned}$$

We first estimate the term with $n = 0$. Using Lemma 2.1 and the estimate (5) with $I = [-1/2, 1/2]$ we have

$$\begin{aligned}&\left|\int_{t_0}^{t} e^{t-s} ds \int_{-1/2}^{+1/2} g_{t-s}(-y) A(s,y)|A(s,y)|^2 dy\right| \\ &\le C_1 \int_{t_0}^{t} e^{t-s} ds \int_{-1/2}^{+1/2} \frac{e^{-y^2/((1+\alpha^2)(t-s))}}{\sqrt{\pi(1+\alpha^2)^{1/2}(t-s)}}\ \|A\|^3_{L^\infty([-1/2,1/2]}dy \\ &\le C_1 e^{t-t_0} \int_{t_0}^{t} \|A\|^3_{L^\infty([-1/2,1/2])} ds \\ &\le C_1 e^{t-t_0} (t-t_0)^{1/4} \left(\frac{1}{\sup_{x\in[-1/2,1/2]}\varphi(x)}\right)^{3/2} \\ &\times \left(\int_{t_0}^{t} \big(2L(s)^2 + 8L(s)L(t_0) + 8CL(s)(s-t_0)\big)\, ds\right)^{3/4}\ .\end{aligned}$$

We now prove a similar estimate for $n \neq 0$. We first observe that since $t-s \le t-t_0 \le 1$ we have for $|y| > |n|-1/2$

$$\big|g_{t-s}(-y)\big| \le C_1 \frac{1}{\sqrt{\pi(1+\alpha^2)^{1/2}(t-s)}} e^{-y^2/((1+\alpha^2)(t-s))}$$

$$\le C_1 \frac{\sqrt{2}}{\sqrt{4\pi(1+\alpha^2)^{1/2}(t-s)}} e^{-y^2/(4(1+\alpha^2)(t-s))} e^{-(|n|-1/2)^2/(4(1+\alpha^2))} \ .$$

We now apply the same estimate as above for each $n \neq 0$, using the function φ_n $(= \varphi(\,\cdot\, - n))$. We finally get

$$\left| \int_{t_0}^{t} e^{t-s} \Big(g_{t-s} * \big(A(s,\,\cdot\,)|A(s,\,\cdot\,)|^2 \big) \Big)(0) ds \right|$$

$$\le 2e^{5/4} C_1 \sum_n e^{-(|n|-1/2)^2/(4(1+\alpha^2))} \left(\frac{1}{\sup_{x\in[-1/2,1/2]} \varphi(x)} \right)^{3/2}$$

$$\times \left(\int_{t_0}^{t} \Big(2\tilde{L}(s)^2 + 8\tilde{L}(s)\tilde{L}(t_0) + 8C\tilde{L}(s)(s-t_0) \Big) ds \right)^{3/4} \ .$$

We can now combine this estimate with estimate (7) in (6) to get

$$|A(t,0)| \le \min\left\{ C_1 e^{t-t_0} \|A(t_0,\,\cdot\,)\|_{L^\infty} \, , \, e^{t-t_0} C_2 \left(\frac{1}{\sqrt{t-t_0}} + \sqrt{t-t_0} \right) \tilde{L}(t_0) \right\}$$

$$+ 2e^{5/4} C_1 \sum_n e^{-(|n|-1/2)^2/(4(1+\alpha^2))} \left(\frac{1}{\sup_{x\in[-1/2,1/2]} \varphi(x)} \right)^{3/2}$$

$$\times \left(\int_{t_0}^{t} \Big(2\tilde{L}(s)^2 + 8\tilde{L}(s)\tilde{L}(t_0) + 8C\tilde{L}(s)(s-t_0) \Big) ds \right)^{3/4} \ .$$

Since the cGL equation is invariant by translation, the same estimate hold for any point (using the adequate translate of the function φ). Therefore we have

$$\|A(t,\,\cdot\,)\|_{L^\infty}$$

$$\le \min\left\{ C_1 e^{t-t_0} \|A(t_0,\,\cdot\,)\|_{L^\infty} \, , \, e^{t-t_0} C_2 \left(\frac{1}{\sqrt{t-t_0}} + \sqrt{t-t_0} \right) \tilde{L}(t_0) \right\}$$

$$+ 2e^{5/4} C_1 \sum_n e^{-(|n|-1/2)^2/(4(1+\alpha^2))} \left(\frac{1}{\sup_{x\in[-1/2,1/2]} \varphi(x)} \right)^{3/2}$$

$$\times \left(\int_{t_0}^{t} \Big(2\tilde{L}(s)^2 + 8\tilde{L}(s)\tilde{L}(t_0) + 8C\tilde{L}(s)(s-t_0) \Big) ds \right)^{3/4} \ . \qquad (8)$$

This is the kind of estimate we were looking for. For t large, the right hand side only involves the function $\tilde{L}(t)$ for which we have an a-priori control (see

(4)). The estimate is bad for t near t_0 and there we can get a better estimate using the L^∞ norm of $A(t_0,\,\cdot\,)$. Combining this estimate with Theorem 2.1 we have proved the following result.

Theorem 2.2. *The solution $A(t,\,\cdot\,)$ of (1) with initial condition $A_0 \in C^0_{b,u}$ exists for all (positive) times. Moreover there exists $M_0 = M_0(\alpha,\beta) > 1$ such that for any $A_0 \in C^0_{b,u}$*

$$\sup_{t>0} \big\|A(t,\,\cdot\,)\big\|_{L^\infty} \le M_0\big(1 + \big\|A(0,\,\cdot\,)\big\|^4_{L^\infty}\big) ,$$

and there exists $T_0 = T_0(\alpha,\beta,A_0) > 0$ (finite) such that

$$\sup_{t>T_0} \big\|A(t,\,\cdot\,)\big\|_{L^\infty} \le M_0 .$$

Proof. For $t \le 1$, the first bound is obvious from inequality (8) using $t_0 = 0$. For $t > 1$, let $t_0 = \lfloor n \rfloor$. The two bounds follow at once from (8) and (4). □

Exercise 2.4. Prove the analogous result in a bounded domain with periodic boundary conditions.

The case $d > 1$ can be controlled along the same lines but with some adapted estimates. We refer to the literature for the proofs (see for example [9] and [21]).

2.2 Regularity of the solutions.

An extension of the above method allows to prove regularity of the solutions. For example one can show that for any initial condition $A_0 \in C^0_{b,u}$, the corresponding solution $A(t,x)$ belongs to $C^0_{b,u}$ for any $t > 0$. However one can prove an even better result which we now formulate for the space dimension d.

Theorem 2.3. *There exists $\delta = \delta(\alpha,\beta) > 0$ and $\tilde M = \tilde M(\alpha,\beta) > 0$ (finite) such that for any $A_0 \in C^0_{b,u}$ there exists $T = T(\alpha,\beta,A_0) > 0$ (finite) such that for any $t > T$ the function $A(t,\,\cdot\,)$ can be extended to a function analytic in the tube*

$$\mathcal{T}_\delta = \left\{ (z_1,\ldots,z_d) \in \mathbf{C}^d \;\middle|\; \sup_{1\le j\le d} \big|\Im z_j\big| \le \delta \right\}$$

where it satisfies

$$\sup_{\underline{z}\in\mathcal{T}_\delta} \big|A(t,\underline{z})\big| \le \tilde M .$$

Remark 2.1. Outside of $\mathbf{R}^d$, the function $A(t,\underline{z})$ does not satisfy equation (1), but it satisfies another equation (in fact a system). This is because the cGL equation is not analytic (it contains $|A|$) and one has to decompose into the real and imaginary parts to obtain an analytic (polynomial) system of evolution equations. is

Exercise 2.5. There exist stationary solutions which are not entire functions. For example in the one dimensional case with $\alpha = \beta = 0$, the reader is invited to check that

$$u(t,x) = \tan\left(\frac{x}{\sqrt{2}}\right)$$

is a solution. Since it does not depend on time it is called a stationary solution. Moreover, this function is analytic in a strip around the real axis. The reader can check that it has singularities at a finite distance of the real axis.

We are not going to prove the above Theorem in its full generality but give a lighter proof for the space dimension one and for the equation

$$\partial_t u = u'' + u - u^3 \ . \tag{9}$$

We refer to the literature for the general case (see for example [8] [31]).

To prove Theorem 2.3, we are going to prove that there is a finite constant $\Gamma > 0$ and a converging increasing sequence of times $t_1, t_2, \ldots$ (depending on the solution) such that for any $t > t_k$ we have

$$\sup_{t>t_k} \left\|\partial_x^k u(t,\,\cdot\,)\right\| \leq \frac{\Gamma^k k!}{(1+k)^2} \ . \tag{10}$$

Theorem 2.3 follows at once from this estimation. Indeed, for any $t > t_\infty = \lim_{k\to\infty} t_k$ and for any $x \in \mathbf{R}$, the Taylor series in x of $u(t,\,\cdot\,)$ converges in a (complex) disk of radius $1/\Gamma$. It now remains to prove the estimate (10). The proof is recursive. For $k = 0$, this is the uniform L^∞ estimate proved in Theorem 2.2. Let $k \geq 1$, and assume the bound has been proved for $j = 0, \ldots, k-1$. From Lemma 2.1 (note that in this case the function g_t is a Gaussian), we have for any $t \geq t_{k-1}$ and any $s > 0$

$$u(t+s,\,\cdot\,) = e^s g_s * u(t,\,\cdot\,) - \int_0^s e^{s-\tau} g_{s-\tau} * u^3(t+\tau,\,\cdot\,) d\tau \ .$$

The right hand side is k times differentiable, and we can apply one differential on the function $g_{s-\tau}$, the other $k-1$ on u^3. The scrupulous reader is invited to check that one can exchange the integration in time and the differentiations. One gets

$$\partial_x^k u(t+s,\,\cdot\,) = e^s g_s' * \partial_x^{k-1} u(t,\,\cdot\,) - \int_0^s e^{s-\tau} g_{s-\tau}' * \partial_x^{k-1}\big(u^3(t+\tau,\,\cdot\,)\big) d\tau \ . \tag{11}$$

We are now going to bound separately the two terms on the right hand side. We take $s = 1/k^2$ and $t_k = t_{k-1} + 1/k^2$. For $t \geq t_{k-1}$, we have by Young's inequality

$$\left\|e^s g_s' * \partial_x^{k-1} u(t,\,\cdot\,)\right\|_{L^\infty} \leq e \left\|g_{1/k^2}'\right\|_{L^1} \left\|\partial_x^{k-1} u(t,\,\cdot\,)\right\|_{L^\infty}$$

and from Lemma 2.1 and the recursion assumption this is smaller than

$$\frac{ek}{\sqrt{\pi}}\frac{\Gamma^{k-1}(k-1)!}{k^2} \le \frac{4e\Gamma^{-1}}{\sqrt{\pi}}\frac{\Gamma^k(k)!}{(1+k)^2} \le \frac{1}{2}\frac{\Gamma^k(k)!}{(1+k)^2}\ , \tag{12}$$

if $\Gamma > 8e/\sqrt{\pi}$ which can always be assumed from the beginning. We now estimate the second term in the right hand side of (11). Using again Young's inequality, we get

$$\left\| \int_0^s e^{s-\tau} g'_{s-\tau} * \partial_x^{k-1}\big(u^3(t+\tau,\,\cdot\,)\big)d\tau \right\|_{L^\infty}$$

$$\le e \sup_{0\le\tau\le s} \left\| \partial_x^{k-1}\big(u^3(t+\tau,\,\cdot\,)\big) \right\|_{L^\infty} \int_0^s \left\| g'_{s-\tau} \right\|_{L^1} d\tau$$

and by Lemma 2.1, this is bounded above by

$$\sup_{t>t_{k-1}} \left\| \partial_x^{k-1}\big(u^3(t,\,\cdot\,)\big) \right\|_{L^\infty} \frac{e}{\sqrt{\pi}} \int_0^s \frac{d\tau}{\sqrt{\tau}} \le \frac{2e}{\sqrt{\pi}} \sup_{t>t_{k-1}} \left\| \partial_x^{k-1}\big(u^3(t,\,\cdot\,)\big) \right\|_{L^\infty}\ .$$

It is left to the reader to prove the following identity

$$\partial_x^{k-1}\big(u^3(t,\,\cdot\,)\big) = (k-1)! \sum_{\substack{l_1+l_2+l_3=k-1 \\ l_1,l_2,l_3\ge 0}} \frac{\partial_x^{l_1}u(t,\,\cdot\,)\partial_x^{l_2}u(t,\,\cdot\,)\partial_x^{l_3}u(t,\,\cdot\,)}{l_1!\ l_2!\ l_3!}\ .$$

Therefore using the recursion assumption we get

$$\left\| \partial_x^{k-1}\big(u^3(t,\,\cdot\,)\big) \right\|_{L^\infty} \le (k-1)! \sum_{\substack{l_1+l_2+l_3=k-1 \\ l_1,l_2,l_3\ge 0}} \frac{\Gamma^{l_1}}{(1+l_1)^2}\frac{\Gamma^{l_2}}{(1+l_2)^2}\frac{\Gamma^{l_3}}{(1+l_3)^2}$$

$$= (k-1)!\Gamma^{k-1} \sum_{\substack{l_1+l_2+l_3=k-1 \\ l_1,l_2,l_3\ge 0}} \frac{1}{(1+l_1)^2}\frac{1}{(1+l_2)^2}\frac{1}{(1+l_3)^2}\ .$$

We now observe that since $l_1 + l_2 + l_3 = k - 1$, and the l's are non negative, at least one must be larger than or equal to $k/4$. Therefore

$$\sum_{\substack{l_1+l_2+l_3=k-1 \\ l_1,l_2,l_3\ge 0}} \frac{1}{(1+l_1)^2}\frac{1}{(1+l_2)^2}\frac{1}{(1+l_3)^2}$$

$$\le 3\left(\sum_{l\ge 0}\frac{1}{(1+l)^2}\right)^2 \sum_{l\ge k/4}\frac{1}{(1+l)^2} \le \frac{C}{(1+k)}\ ,$$

where C is a positive constant independent of k. We therefore get

$$\left\| \int_0^s e^{s-\tau} g'_{s-\tau} * \partial_x^{k-1}\big(u^3(t+\tau,\,\cdot\,)\big)d\tau \right\|_{L^\infty} \le \frac{2eC\Gamma^{-1}}{\sqrt{\pi}}\frac{1+k}{k}\frac{\Gamma^k k!}{(1+k)^2}.$$

By increasing eventually the value of Γ (this does not spoil the previous estimate), we can assume that (since $k \geq 1$)

$$\left\| \int_0^s e^{s-\tau} g'_{s-\tau} * \partial_x^{k-1}\big(u^3(t+\tau,\,\cdot\,)\big) d\tau \right\|_{L^\infty} \leq \frac{1}{2} \frac{\Gamma^k k!}{(1+k)^2} .$$

Combining this estimate with estimate (12) we get the estimate at level k. The recursive argument is therefore complete and we have proved Theorem 2.3.

3 Attracting sets.

A dynamical system in a finite dimensional phase space Ω (often a manifold) and continuous time is defined by a (regular) vector field to which is associated a flow of evolution. This is a one parameter family $\Phi_{t\in\mathbf{R}}$ of diffeomorphisms of the phase space satisfying $\Phi_0 = \text{Identity}$ and for any reals s and t, $\Phi_{t+s} = \Phi_t \circ \Phi_s$. If x is the state of the system at time $t = 0$ (a point in the phase space), then the state of the system at time t is the point $\Phi_t(x)$.

A subset $\mathscr{A}$ of Ω is called an attracting set if it is

1) Invariant ($\Phi_t(\mathscr{A}) = \mathscr{A}$ for any $t \in \mathrm{R}$)

2) Compact

3) There exists a neighborhood V of $\mathscr{A}$ such that for any neighborhood U of $\mathscr{A}$ there is a time $T = T(V, U)$ such that for any $t > T$ we have $\Phi_t(V) \subset U$.

The last property reflects the fact that the orbit of any initial condition in V converges to $\mathscr{A}$. In other words, the asymptotic behaviour of such orbits is described by the orbits in $\mathscr{A}$. The set of accumulation points of an orbit (the so called ω limit set) is contained in $\mathscr{A}$. In practice, one wants to choose V as large as possible (the basin of $\mathscr{A}$), and $\mathscr{A}$ as small as possible.

For unbounded domains, it is in general not possible to define an evolution flow, but only an evolution semi-flow, i.e. a one parameter family (group) $\Phi_{t\in\mathbf{R}^+}$ defined only for positive times but satisfying the same properties as above. The definition of attracting sets has also to be revised. The compactness condition should be abandoned and the third condition of convergence is too strong. A good definition of attractors for such systems was first proposed by Feireisl [19] and then generalised by Mielke and Schneider [26], see also [22]). It is based on a pragmatic observational point of view. One defines the attracting set from observations in finite windows, but the attracting set does not depend of the windows.

We will now state the required hypothesis and the definitions in a general situation. We will discuss as an example the application to the Ginzburg-Landau equation studied in the previous section. In the construction of the global time evolution, we have used function (Banach) spaces with weights. We will denote by $\mathscr{B}$ such a function space. For the Ginzburg-Landau equation, this can be the Banach space

$$H^1_\varphi = \left\{ A \,\middle|\, \int \varphi(x)\big[|A(x)|^2 + |A'(x)|^2\big]\,dx < \infty \right\}$$

with $\varphi(x) = 1/(1+x^2)$. Note that this space is not contained in L^∞.

Exercise 3.1. Verify that equipped with the norm

$$\|A\|_{H^1_\varphi} = \left(\int \varphi(x)\big[|A(x)|^2 + |A'(x)|^2\big]\,dx \right)^{1/2} \;,$$

the space H^1_φ is a Banach space.

For any $y \in \mathbf{R}^d$, we define the translation operator T_y by

$$T_y A(x) = A(x_y) \;.$$

Hypothesis 3.1. For any $y \in \mathbf{R}^d$, the translation operator T_y is bounded in $\mathscr{B}$.

Exercise 3.2. Verify that the translation operators T_y are bounded in $\mathscr{B}H^1_\varphi$. Hint: show that for a fixed y, $\sup_x \varphi(x+y)/\varphi(x)$ is finite.

One can now define the "uniform" space $\mathscr{B}_u$ associated to the space $\mathscr{B}$ by

$$\mathscr{B}_u = \left\{ A \in \mathscr{B} \,\middle|\, \sup_{y \in \mathbf{R}^d} \|T_y A\|_{\mathscr{B}} < \infty \right\} \;.$$

In general, $\mathscr{B}_u$ is much smaller than $\mathscr{B}$.

Exercise 3.3. Verify that equipped with the norm

$$\|A\|_{\mathscr{B}_u} = \sup_{y \in \mathbf{R}} \|T_y A\|_{\mathscr{B}}$$

the space $\mathscr{B}_u$ is a Banach space.

Note that if φ has compact support (which however is not a very convenient choice for the estimates of section 2), the H^1_φ norm measures in some sense the size of the function in a window which is the support of H^1_φ. On the other hand, the norm of $H^1_{\varphi,u}$ measures in some sense the maximal size.

We now state a Lemma relating the topologies of $\mathscr{B}$ and $\mathscr{B}_u$.

Lemma 3.1. *If $\mathscr{K}$ is a bounded subset of $\mathscr{B}_u$, then its closure in $\mathscr{B}$ is a bounded subset of $\mathscr{B}_u$.*

Proof. Let M denote the radius of a ball containing $\mathscr{K}$ in $\mathscr{B}_u$ (such a ball exists since $\mathscr{K}$ is bounded). Let w be a point in $\overline{\mathscr{K}}^{\mathscr{B}}$ (the closure of $\mathscr{K}$ in $\mathscr{B}$). Then there is a sequence $(u_n) \subset \mathscr{K}$ such that $u_n \to w$ in $\mathscr{B}$ when n tends to infinity. Let $y \in \mathbf{R}^d$. Since the operator T_y is continuous in $\mathscr{B}$, we have $T_y u_n \to T_y w$ when n tends to infinity. Therefore, since

$$\|T_y w\|_{\mathscr{B}} \le \|T_y(w - u_n)\|_{\mathscr{B}} + \|T_y u_n\|_{\mathscr{B}} \le \|T_y(w - u_n)\|_{\mathscr{B}} + M$$

we have for any $y \in \mathbf{R}^d$

$$\|T_y w\|_{\mathscr{B}} \le M$$

and the Lemma follows from the definition of $\mathscr{B}_u$. □

The next hypothesis establishes a link between the time evolution and the function space.

Hypothesis 3.2. The evolution semi-flow Φ_t is continuous in $\mathscr{B}_u$ and commutes with translations. Moreover, if $\big(w_j\big)$ is a bounded sequence in $\mathscr{B}_u$ converging in $\mathscr{B}$ to $w \in \mathscr{B}_u$, then for any $t \ge 0$, $\Phi_t\big(w_j\big)$ converges in $\mathscr{B}$ to $\Phi_t(w)$.

Note that Lemma 3.1, ensures $w \in \mathscr{B}_u$.

Exercise 3.4. Check this hypothesis for the semi-flow of evolution associated to the Ginzburg-Landau equation and the space $\mathscr{B} = H^1_\varphi$.

Let $\mathscr{D} \subset \mathscr{B}_u$ be a bounded (non empty) subset which is translation invariant ($T_y(\mathscr{D}) \subset \mathscr{D}$ for any y) and invariant by the time evolution ($\Phi_t(\mathscr{D}) \subset \mathscr{D}$ for any non negative t). We can now state and prove the following result (see [26]).

Theorem 3.1. *Assume there exists a positive number $t_0 = t_0(\mathscr{D})$ such that $\Phi_{t_0}(\mathscr{D})$ is precompact in $\mathscr{B}$ and $\mathscr{D}$ is absorbing the bounded sets in $\mathscr{B}_u$ (namely for any bounded set $\mathscr{C} \in \mathscr{B}_u$ there a positive number $T = T(\mathscr{C}, \mathscr{D})$ such that for any $t > T$ we have $\Phi_t(\mathscr{C}) \in \mathscr{D}$). Then there exists a set $\mathscr{A} \subset \mathscr{B}_u$ such that*

1) $\mathscr{A}$ is non empty closed and bounded in $\mathscr{B}_u$, compact in $\mathscr{B}$.

2) $\mathscr{A}$ is invariant by $\big(\Phi_t\big)_{t\in\mathbf{R}^+}$ and $\big(T_y\big)_{y\in\mathbf{R}^d}$.

3)

$$\mathrm{dist}_{\mathscr{B}_u}\big(\Phi_t(\mathscr{D}), \mathscr{A}\big) = \sup_{B\in\mathscr{D}} \inf_{A\in\mathscr{A}} \big\|\Phi_t(B) - A\big\|_{\mathscr{B}_u} \overset{t\to\infty}{\longrightarrow} 0 \; .$$

In particular, for any $B \in \mathscr{B}_u$ we have

$$\lim_{t\to\infty} \mathrm{dist}_{\mathscr{B}_u}\big(\Phi_t(B), \mathscr{A}\big) = 0 \; .$$

4) If φ_1 and φ_2 are such that the spaces $\mathscr{B}_{u,\varphi_1}$ and $\mathscr{B}_{u,\varphi_2}$ are the same spaces with the same topology (but the norms are in general different) then the corresponding sets $\mathscr{A}_1$ and $\mathscr{A}_2$ are identical.

5) The set $\mathscr{A}$ does not depend on the set $\mathscr{D}$ with the above properties.

The set $\mathscr{A}$ above is often called a globally attracting set.

Before giving the proof of this Theorem, we check that the hypothesis hold for the cGL equation. Recall that $\mathscr{B} = H^1_\varphi$ and Φ_t is the map from the (function) initial condition $A_0(\,\cdot\,)$ to the function $A(t,\,\cdot\,)$, the solution at time. We have seen in section 2 that this is indeed a continuous semi-group of evolution and since the equation is homogeneous, this semi-group is translation invariant. We also know from section 2 that if D is a large enough ball in $H^1_{u,\varphi}$, it is absorbing and also absorbing itself in a finite time in the following sense. There is a finite number $T_0 = T_0(D) > 0$ such that for any $A \in D$ and any $t > T_0$ we have $\Phi_t(A) \subset D$. Therefore the set

$$\mathscr{D} = \bigcup_{t=0}^{t=T_0} \Phi_t(D)$$

is bounded, invariant and absorbing. We also know from section 2 that the semi flow regularises. Namely, there is a finite positive number $T_1 = T_1(D)$ and a bounded set $D' \in H^2_{u,\varphi}$ such that for any $t > T_1$, $\Phi_t(D) \subset D'$. This immediately implies that for $t > T_1+T_0$, $\Phi_t(\mathscr{D})$ is precompact in H^1_φ. We can therefore apply the above Theorem to ensure the existence of the attracting set $\mathscr{A}$ with the required properties.

Exercise 3.5. Consider the functions $\varphi_1 = 1/(1+x^2)$ and $\varphi_2 = 1/(1+x^4)$. Show that $H^1_{\varphi_1} \neq H^1_{\varphi_2}$. Show however that as topological spaces $H^1_{u,\varphi_1} = H^1_{u,\varphi_2}$.

We now prove the Theorem.

Proof. Let

$$\mathscr{A} = \bigcap_{t\geq 0} \overline{\Phi_t(\mathscr{D})}^{\mathscr{B}} \ .$$

The family $\left(\overline{\Phi_t(\mathscr{D})}^{\mathscr{B}}\right)_{t\in\mathbf{R}^+}$ is decreasing composed of closed bounded (by Lemma 3.1) non-empty subsets of $\mathscr{B}_u$, compact in $\mathscr{B}$ for any $t > t_0$, and translation invariant. Therefore $\mathscr{A}$ is a non empty closed bounded subset of $\mathscr{B}_u$, compact in $\mathscr{B}$.

We now claim that $\mathscr{A}$ is invariant by the time evolution. Indeed, if $v = \Phi_t(u)$ with $u \in \mathscr{A}$, then $u \in \overline{\mathscr{D}}^{\mathscr{B}}$ and hence there is a sequence $(u_n) \subset \mathscr{D}$ such that in $\mathscr{B}$, we have $u_n \to u$ when n tends to infinity. Therefore, by hypothesis 3.2 we have

$$v = \Phi_t(u) \in \bigcap_{\tau\geq t} \overline{\Phi_\tau(\mathscr{D})}^{\mathscr{B}} \ .$$

Since $\mathscr{D}$ is invariant ($\Phi_s(\mathscr{D})\subset\mathscr{D}$ for any $s\ge 0$) we have

$$\bigcap_{\tau\ge t}\overline{\Phi_\tau(\mathscr{D})}^{\mathscr{B}}\subset\bigcap_{\tau\ge 0}\overline{\Phi_\tau(\mathscr{D})}^{\mathscr{B}}$$

which immediately implies $v\in\mathscr{A}$. In other words we have proved that for any $t\ge 0$, $\Phi_t(\mathscr{A})\subset\mathscr{A}$.

We now show that for any $t\ge 0$, $\Phi_t(\mathscr{A})=\mathscr{A}$. Let $v\in\mathscr{A}$, then there exists a sequence $(A_n)\subset\mathscr{D}$ such that $\Phi_n(A_n)$ converges to v in $\mathscr{B}$. Assume now n is large enough so that $n>t+t_0$. Then $\Phi_n(A_n)=\Phi_t(\Phi_{n-t}(A_n))$ and since $n-t>t_0$ we have

$$\Phi_{n-t}(A_n)\in\Phi_{n-t}(\mathscr{D})\subset\Phi_{t_0}(\mathscr{D})$$

which is precompact in $\mathscr{B}$. Therefore, there exists $w\in\mathscr{B}$ and a sequence (n_j) of integers such that (in the topology of $\mathscr{B}$)

$$\lim_{j\to\infty}\Phi_{n_j-t}\big(A_{n_j}\big)=w\ .$$

Since $\Phi_{t_0}(\mathscr{D})$ is a bounded set in $\mathscr{B}_u$, it follows from Lemma 3.1 that $w\in\mathscr{B}_u$, and in $\mathscr{B}$ we have from hypothesis 3.2

$$v=\lim_{j\to\infty}\Phi_{n_j}\big(A_{n_j}\big)=\lim_{j\to\infty}\Phi_t\big(\Phi_{n_j-t}\big(A_{n_j}\big)\big)=\Phi_t(w)\ .$$

Since $\overline{\Phi_t(\mathscr{D})}^{\mathscr{B}}$ is invariant by translation, the same is true for $\mathscr{A}$. This finishes the proof of *2*). To prove *3*), we first establish the result in $\mathscr{B}$. Assume that $\mathrm{dist}_{\mathscr{B}}\big(\Phi_t(\mathscr{D}),\mathscr{A}\big)$ does not tend to zero when t tends to infinity. From the definition of the distance between sets, we conclude that there is a number $\delta>0$, a diverging sequence of positive numbers $\big(t_n\big)$, and a sequence $\big(A_n\big)$ contained in $\mathscr{D}$ such that for any $B\in\mathscr{A}$ we have

$$\big\|\Phi_{t_n}\big(A_n\big)-B\big\|_{\mathscr{B}}>\delta\ .$$

For $t_n>t_0$, $\Phi_{t_n}\big(A_n\big)\subset\Phi_{t_0}(\mathscr{D})$ which is compact in $\mathscr{B}$ by hypothesis. Therefore, we can find a diverging sequence of integers $\big(n_j\big)$ and $w\in\mathscr{B}$ such that in $\mathscr{B}$ we have

$$\lim_{j\to\infty}\Phi_{t_{n_j}}\big(A_{n_j}\big)=w\ .$$

It follows as above that $w\in\mathscr{A}$ which is a contradiction.

We now come to the convergence in $\mathscr{B}_u$. From the definition of the norm in $\mathscr{B}_u$ and by hypothesis 3.2 we have

$$\mathrm{dist}_{\mathscr{B}_u}\big(\Phi_t(\mathscr{D}),\mathscr{A}\big)=\sup_{B\in\mathscr{D}}\inf_{A\in\mathscr{A}}\sup_y\big\|T_y\big(\Phi_t(B)\big)-T_y(A)\big\|_{\mathscr{B}}$$

$$= \sup_{y} \sup_{B\in\mathscr{D}} \inf_{A\in\mathscr{A}} \left\| \Phi_t(T_y(B)) - T_y(A) \right\|_{\mathscr{B}_u}$$

Since T_y is invertible (with inverse T_{-y}), and $\mathscr{D}$ and $\mathscr{A}$ are translation invariant, we have

$$\mathrm{dist}_{\mathscr{B}_u}\big(\Phi_t(\mathscr{D}), \mathscr{A}\big) = \mathrm{dist}_{\mathscr{B}}\big(\Phi_t(\mathscr{D}), \mathscr{A}\big) \ ,$$

and the first part of *3*) is now proved. Since $\mathscr{D}$ is absorbing we get the second part of *3*) as an immediate consequence.

We now come to the last two points in the Theorem. Assume $\mathscr{B}_1$ and $\mathscr{B}_2$ are two (different) Banach spaces satisfying all the above hypothesis and which lead to the same topological spaces $\mathscr{B}_{1,u}$ and $\mathscr{B}_{2,u}$. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ denote the two globally attracting sets obtained from the same bounded set $\mathscr{D}$. In the space $\mathscr{B}_2$, we have

$$\mathrm{dist}_{\mathscr{B}_{2,u}}\big(\Phi_t(\mathscr{A}_1), \mathscr{A}\big) \le \mathrm{dist}_{\mathscr{B}_{2,u}}\big(\Phi_t(\mathscr{D}), \mathscr{A}\big) \overset{t\to\infty}{\longrightarrow} 0 \ .$$

Since $\mathscr{A}_1$ is invariant by the time evolution this implies $\mathrm{dist}_{\mathscr{B}_u}\big(\mathscr{A}_1, \mathscr{A}_2\big) = 0$ and hence $\mathscr{A}_1 \subset \mathscr{A}_2$. By a similar argument $\mathscr{A}_2 \subset \mathscr{A}_1$ and *4*) is proved.

Finally let $\mathscr{D}'$ be another set with the same properties as $\mathscr{D}$. Since $\mathscr{D}$ is bounded, it is absorbed in a finite time by $\mathscr{D}'$. Therefore since $\mathscr{A}$ is invariant we have $\mathscr{A} \subset \mathscr{D}'$. The reverse inclusion implies easily the last assertion *5*). □

Exercise 3.6. Prove that for any $t < 0$ Φ_t is well defined on $\mathscr{A}$ and $\Phi_t(\mathscr{A}) = \mathscr{A}$ (see the proof of *2*) in the above Theorem). .

It is in general impossible to describe explicitly the globally attracting set except in trivial cases. For example for the real Ginzburg-Landau equation

$$\partial_t u = u'' - u - u^3$$

in the space $H^1_{u,\varphi}$ the globally attracting set is $\mathscr{A} = \{0\}$.

Exercise 3.7. Prove this assertion using a local energy estimate.

Consider now the evolution equation

$$\partial_t u = u'' + u - u^3 \ , \tag{13}$$

where u is real. Its globally attracting set is much richer. First of all one can look for (bounded) stationary solutions. This amounts to solving the ordinary differential equation $u'' + u - u^3 = 0$ which has a mechanical interpretation left to the reader. There are three constant solutions $u = 0$ and $u = \pm 1$. There are also non constant solutions like $u = \pm\tanh((x - x_0)/\sqrt{2})$ and some periodic in space stationary solutions (elliptic functions).

Exercise 3.8. Use the mechanical analogy (Newton's equation) to qualitatively derive the above assertions.

Another important class of solutions contained in the globally attracting set are called fronts. They are in some sense the analog of the solitons: when viewed in a frame moving at an adequate speed, these solutions are stationary. In other words, these solutions are of the form $u(t,x) = v(x-ct)$ where c is the velocity and v the profile of the front. Note that for $c = 0$ one recovers the stationary solutions. In order to investigate if such solutions exist, one plugs the ansatz $u(t,x) = v(x-ct)$ into the evolution equation. Here we obtain an equation for v

$$v'' + cv' + v - v^3 = 0 \ ,$$

which depends on the parameter c. There are many solutions for this equation. Let us consider those which tend to zero for x tending to $+\infty$ and to $+1$ for x tending to $-\infty$ (one can prove that such solutions exist, see [12]). Such solutions have an interesting interpretation, they model the invasion of the unstable phase $u = 0$ by the stable phase $u = 1$. There are (real) solutions for all the positive values of c. For $c \geq 2$ the solutions are positive, they oscillate for $0 < c < 2$. Let us now show that these functions are on the globally attracting set. More precisely we will show that for any $a \in \mathbf{R}$, the function $f(x) = v(x-a)$ belongs to the globally attracting set. For this purpose it is enough to construct for any t a function g_t belonging to $\mathscr{D}$ such that $\Phi_t(g_t) = f$. The absorbing set $\mathscr{D}$ is constructed as explained above. We start with a ball D in $H^1_{u,\varphi}$ with diameter large enough so that $f \in \mathscr{D}$, and we take $\mathscr{D} = \cup_{s=0}^{T_0} \Phi_s(D)$ for some $T_0 > 0$ large enough. Let $g_t(x) = v(x-a+ct)$. By translation invariance, we have $g_t \subset \mathscr{D}$ for any t. Since $\Phi_t(g_t)(x) = v(x-ct+a+ct) = f$, we have $f \in \mathscr{A}$.

We have seen here on a particular case a general property of the functions on the globally attracting set, namely if $f \in \mathscr{A}$, for any $t > 0$, there if a function $g_t \in \mathscr{A}$ such that $\Phi_t(g_t) = f$. This is in general not true outside the globally attracting set.

We refer to the (large) literature and in particular to [1], [3], [12], [29], [33], [17] for more on the subject of fronts and related solutions.

4 Instabilities.

As in low dimensional dynamical systems, the dependence of the system upon a parameter is a very important concept. It furnishes a tool to study different levels of complexity of the dynamics through the appearance of the instabilities which are in some sense the analogs of the bifurcations in low dimensional systems. Moreover, as we will see below, the instabilities provide an explanation for the occurrence of the structures and of their properties.

4.1 Instabilities in the Swift-Hohenberg equation.

In order to keep the notations at a minimum of complexity we will study the particular case of the Swift-Hohenberg equation (14). The Swift-Hohenberg equation is the equation for a real field $u(t,x)$ which depends on time and on a one dimensional space variable x. It also depends on a real parameter μ and is given by

$$\partial_t u = -\big(1+\partial_x^2\big)^2 u + \mu u - u^3 \ . \tag{14}$$

It was introduced by Swift and Hohenberg as a simplified model for the study of the instability in the Rayleigh Benard experiment (fluid heated from below). The form of the non linear part is for convenience and not very physical. The solution u represents the velocity of the fluid at position x (this should be thought as a horizontal coordinate, there is a vertical coordinate which does not appear in this caricature of the experiment).

Exercise 4.1. Prove that for any value of μ, the semi-flow of evolution is globally well defined in the spaces discussed in section 2.

The approach described below has been applied to many other more realistic equations, we refer for example to the classical books [5], [15] and the review [14] for details. The first observation is that for any value of the parameter μ, the function $u = 0$ is a solution of the equation. If we think of u as the velocity of a fluid, this means that the fluid is at rest. It is then natural to ask if this situation is stable, namely if some small perturbation (often present in the experiment) will decay or on the opposite amplify, putting the fluid in motion. If we start will a small perturbation, by the continuity of the evolution semi-flow in the initial condition, the solution will remain small for some time and it is therefore reasonable to neglect the non linear part. We obtain the linearised evolution equation given by

$$\partial_t v = -\big(1+\partial_x^2\big)^2 v + \mu v \ . \tag{15}$$

This equation being linear and with constant coefficients can be solved using Fourier transform in space. Indeed one gets for the Fourier transform (in space) $\hat{v}$ the equation

$$\partial_t \hat{v}(t,k) = \Omega(\mu,k)\hat{v}(t,k) \tag{16}$$

where

$$\Omega(\mu,k) = -\big(1-k^2\big)^2 + \mu \ .$$

The solution is of course

$$\hat{v}(t,k) = e^{t\Omega(k)}\hat{v}(0,k) \ .$$

Therefore if for some k we have $\Omega(k) < 0$, the function of time $\hat{v}(t,k)$ tends to zero when t tends to infinity. In particular, when $\mu < 0$, we have $\Omega(k) < 0$ for all k and therefore $\hat{v}(t, \cdot)$ tends to zero. One has to be a little careful about this argument because for the function spaces described in section 2, the Fourier transform $\hat{v}$ is in general not a function but a tempered distribution.

Exercise 4.2. Prove that for any initial condition $v_0 \in L^\infty$ and for any $\mu \leq 0$, the solution of (15) tends to zero in L^∞ (hint: estimate the decay in space and time of the kernel solving the equation, see also the technical lemmas below used in the renormalisation argument).

This is of course only a result in the linear approximation. In the case of the Swift-Hohenberg equation one can do better.

Exercise 4.3. Prove that for any $\mu \leq 0$ and for the space $H^4_{\varphi,u}$ of section 2, the global attracting set reduces to the function $u = 0$ (use adequate energy estimates).

It is now interesting to plot the function $k \to \Omega(\mu, k)$ for different values of μ (see figure 1).

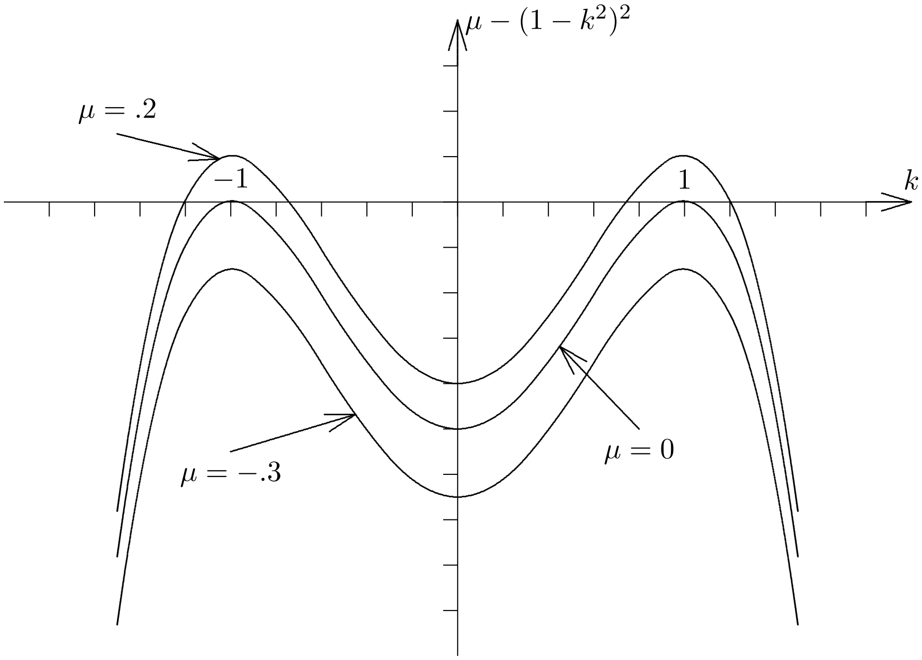

Figure 1: Graph of Ω for different values of μ

We see that for $\mu > 0$ small, there are two symmetrical intervals of values of k for which $\Omega(\mu, k) > 0$. These intervals are more or less centered around the points $k = \pm 1$. If we consider initial conditions whose Fourier transform have support intersecting (the interior of) these intervals, the corresponding solution of the linearised equation will grow exponentially fast. We have found a condition for instability.

It is convenient at this point to emphasise the different findings and compare them with the experimental observations described in the introduction. First we have a threshold of instability, here the critical parameter value $\mu_c = 0$. For $\mu < \mu_c$ all perturbations decay to zero, the stationary solution (fluid at rest) is stable. For $\mu > \mu_c$ some perturbations get (linearly) amplified. They correspond to wave-numbers k about ± 1. In other words, we have found a particular wavelength in the problem, and this will be the wavelength of the structures. This is exactly what is observed in the experiment: a reproducible threshold of instability, at which structures of fixed wavelength appear. The critical wave numbers $k_c = \pm 1$ are those wave numbers which are marginally stable at the critical value of the parameter.

This kind of computations has been performed in many systems and agrees quite well with experimental observations. We refer to [5], [15] and [14] for more examples.

For later purposes, we now consider in more detail the case of positive small μ. The zone of positive wave-numbers where $\Omega(\mu, k) > 0$ is obviously given by

$$|1-k| < \frac{\sqrt{\mu}}{2} + \mathscr{O}(\mu) \ .$$

It is therefore useful to introduce the positive parameter $\epsilon = \sqrt{\mu}$, and we get

$$|1-k| < \frac{\epsilon}{2} + \mathscr{O}(\epsilon^2) \ .$$

Similarly, for negative k the unstable zone of wave numbers is given by $|1+k| < \epsilon/2 + \mathscr{O}(\epsilon^2)$. We also observe that a new large space scale has appeared in the problem, namely $1/\epsilon$. This is because we have a small scale ϵ for the wave numbers besides the critical scale equal to unity (see figure 2).

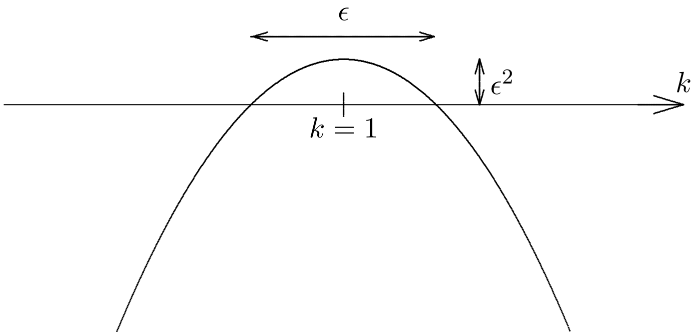

Figure 2: Local behaviour of Ω near one of its maximum.

Before discussing the non linear effects, we can at this point generalise the above approach and introduce the important distinction between absolute and convective instabilities. As already mentioned, it is often interesting to consider dynamical systems with space variables as extensions of the simpler finite dimensional dynamical systems. We can think of the right hand side of equation (14) as some kind of vector field on some infinite dimensional phase space. We assume that we have a stationary solution which may depend on the parameter μ (fixed point), and to investigate its linear stability, we look at the linearised equation around the fixed point. The analog of equation (15) takes the form

$$\frac{dv}{dt} = L_\mu v$$

where for (14), L_μ is the linear differential operator

$$L_\mu v = -\big(1+\partial_x^2\big)^2 v + \mu v \ .$$

One could think of investigating the spectrum of the operator L_μ as a function of μ. This requires first defining an adequate Banach space (for example the spaces considered in section2). We also observe that if the system is homogeneous, this operator commutes with translations, and therefore can be diagonalised using Fourier transform. This leads to equations of the form

$$\frac{d\hat{v}_k}{dt} = L_\mu(k)\hat{v}_k$$

where here $L_\mu(k)$ is a polynomial in k. In more general situations, v is a vector, and $L_\mu(k)$ is a matrix of constant dimension r depending on k and μ. The time evolution of $\hat{v}_k$ will then depend on the spectrum of $L_\mu(k)$. It is important to notice that it is often convenient to work with real valued solutions. If the solution v is complex, one can use the vector composed of the real and imaginary parts, this leads immediately to a two by two matrix $L_\mu(k)$. If we are working with real equations (the entries of the matrix $L_\mu(k)$ are real), the eigenvalues satisfy the following relations $\lambda(\mu,k) = \overline{\lambda}(\mu,-k)$.

Exercise 4.4. Consider the equation

$$\partial_t A = A'' + A - A|A|^2 \ . \tag{17}$$

Show that for any $|k|\le 1$, the function $A_k(x)=\sqrt{1-k^2}e^{ikx}$ is a stationary solution. Consider a small perturbation of A_k of the form $A(t,x)=A_k(x)+e^{ikx}v(t,x)$. Write the system of linearised evolution equations for the real and imaginary parts of v.

For a fixed value of the parameter μ, one should look at the eigenvalues $\lambda_1(\mu,k),\dots,\lambda_r(\mu,k)$ when k varies. We obtain curves in the complex plane parametrised by μ (which may cross each other), and if for all k and $1\le j\le r$ we have $\Re\lambda_j(\mu,k)<0$, then the stationary solution is linearly stable (it is said marginally stable if $\Re\lambda_j(\mu,k)\le 0$ for all j and k and zero is reached).

Exercise 4.5. Draw the curves of $\lambda_1(\mu, k), \ldots, \lambda_r(\mu, k)$ for the Swift-Hohenberg equation (14) linearised around the solution $u = 0$ ($r = 1$ and the curves are degenerate).

Exercise 4.6. Show that for the real Ginzburg-Landau equation (13) the stationary solution $u = \pm 1$ is linearly stable, but the stationary solution $u = 0$ is linearly unstable.

Exercise 4.7. For equation (17), determine the range of k where there is in the linear approximation exponentially growing perturbations of the stationary solution A_k. (Eckhaus instabilities).

Exercise 4.8. Compute $\lambda_1(\mu, k), \lambda_2(\mu, k)$ for the zero solution of the evolution system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x^6 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix} \partial_x^3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \mu - 1 & 0 \\ 0 & \mu - 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} u_1^3 \\ u_2^3 \end{pmatrix}, \tag{18}$$

where u_1 and u_2 are real functions, and μ is a real parameter. Draw the curves of $\lambda_1(\mu, k), \lambda_2(\mu, k)$ in the complex plane for different values of μ. Find the critical value of μ where linear instability appears.

However, when μ changes, the curves may change also. If we start from a stable situation, one may reach a parameter value μ_c where for the first time one (or several) of the curves reach the imaginary axis. Such a curve is often called a critical branch, and there is associated a critical wave number k_c (or several) where the curve touches the imaginary axis. Let j_0 be the index of such a branch. We have by definition $\Re\lambda_{j_0}(\mu_c, k_c) = 0$. We also have $\Re\partial_k\lambda_{j_0}(\mu_c, k_c) = 0$, since otherwise there is a piece of the branch in the right half plane and therefore μ_c cannot be a critical value. We will now make the generic non degeneracy assumption $\Re\partial_k^2\lambda_{j_0}(\mu_c, k_c) < 0$.

Exercise 4.9. Show that if $\Re\partial_k^2\lambda_{j_0}(\mu_c, k_c) > 0$, μ_c cannot be a critical value.

We can now state the following definition.

Definition 4.1. If $c = \Im\partial_k\lambda_{j_0}(\mu_c, k_c) \neq 0$ the instability is said to be convective, in the other case it is called absolute.

Exercise 4.10. Show that the system (18) has a convective instability.

We now explain the important difference between these two types of instabilities. We will now assume that the instability is non degenerate, namely that $b = \Re\partial_\mu\lambda_{j_0}(\mu_c, k_c) \neq 0$. For definiteness we will assume that this quantity is positive (the case where it is negative is treated similarly and left to the reader). If we denote by γ the quantity $\Im\lambda_{j_0}(\mu_c, k_c)$, and $a = -\Re\partial_k^2\lambda_{j_0}(\mu_c, k_c) > 0$ we have for k near k_c and μ near μ_c

$$\lambda_{j_0}(\mu, k) = i\gamma + b(\mu - \mu_c) - a\frac{(k - k_c)^2}{2} + ic(k - k_c) + \text{h.o.t.}$$

where the correction (h.o.t.) is quadratic in $\mu - \mu_c$ and cubic in $k - k_c$. Let us now consider an initial perturbation $v(0, x)$ such that its Fourier transform $\hat{v}(0, k)$ has support only near k_c. For simplicity, we assume $r = 1$ (recall that r is the dimension of the solution) although the extension of the argument to any finite r is immediate. Neglecting the corrections to $\lambda_{j_0}(\mu, k)$ we can write

$$v(t, x) = \int e^{t[i\gamma + b(\mu-\mu_c) - a(k-k_c)^2/2 + ic(k-k_c)]} e^{ikx} \hat{v}(0, k) dk \ .$$

To see more clearly what this function looks like, we first consider it in a frame moving at velocity c. Namely we consider the function

$$w(t, y) = v(t, y - ct) = e^{t[i\gamma + b(\mu-\mu_c) - ick_c]} \int e^{-at(k-k_c)^2/2} e^{iky} \hat{v}(0, k) dk$$

$$= e^{it\gamma + bt(\mu-\mu_c) - ick_c t + ik_c y} \int e^{-at(k-k_c)^2/2} e^{i(k-k_c)y} \hat{v}(0, k) dk$$

For t large, one can use a steepest descent argument (see [24] and [23]) which leads to the following asymptotic

$$w(t, y) =$$

$$e^{it\gamma + bt(\mu-\mu_c) + ik_c(y-ct)} \hat{v}(0, 0) \int e^{-at(k-k_c)^2/2} e^{i(k-k_c)y} dk \ + h.o.t.$$

$$= e^{it(\gamma - ck_c) + bt(\mu-\mu_c) + ik_c y} \hat{v}(0, 0) \frac{e^{-y^2/(2at)}}{\sqrt{2\pi a t}} \ + \text{h.o.t.}$$

We conclude that in the moving frame

1. There is an oscillation in t (provided $\gamma - ck_c \neq 0$). This is the time scale of the structure.

2. There is an oscillation in y of period $2\pi/k_c$ this is the space scale of the structure. In fact at this approximation, the solution develops to a periodic function in space.

3. If $|y| >> t\sqrt{2ab(\mu - \mu_c)}$, the solution becomes very small. In other words, in the frame moving at velocity c, the instability is a linearly expanding bubble inside which we see an oscillation of spatial period $2\pi/k_c$ with exponentially growing amplitude.

We are however more interested in the interpretation in a laboratory frame (fixed observer) where the solution is

$$v(t, x) = e^{it\gamma + bt(\mu-\mu_c) + ik_c x} \hat{v}(0, 0) \frac{e^{-(x-ct)^2/(2at)}}{\sqrt{2\pi a t}} \ + \text{h.o.t.}$$

We conclude that in the laboratory frame

1. There is an oscillation in t (provided $\gamma \neq 0$).
2. There is an oscillation in x of period $2\pi/k_c$ this is the spatial scale of the structure. The structure is the harmonic wave $e^{i(\gamma t+k_c x)}$ for $(\mu-\mu_c)$ small.
3. If $|x-ct| >> t\sqrt{2ab(\mu-\mu_c)}$, the solution becomes very small. In other words, the instability is a bubble moving at velocity c, expanding at velocity $\sqrt{2ab(\mu-\mu_c)}$ inside which we see an oscillation of spacial period $2\pi/k_c$ and time period $2\pi/\gamma$ with exponentially growing amplitude at rate $b(\mu-\mu_c)$ (the spatial expansion of this bubble is related to the fronts discussed in the previous section).

If $c \neq 0$ (convective instability), for small $\mu-\mu_c > 0$, we have

$$\sqrt{2ab(\mu-\mu_c)} < |c| \ .$$

Therefore in the laboratory frame, a fixed observer may see such a bubble passing in front of him, but afterward, the solution seems to relax to the stationary state (unless another such perturbation is excited). The instability is convected at velocity c and its extension grows but at a much slower rate. If $c = 0$, the instability does not move, it simply expands and invades larger and larger domains. This is the absolute case.

If one consider the one parameter family of fronts for the equation (13) discussed in the previous section, one can show that they are convectively unstable except for the velocity $c = 2$. We refer to [1], [3], [12], [29] for more details.

4.2 Amplitude equation.

Up to now we have discussed the linearised equation around a stationary solution. Of course if an instability is growing exponentially fast, after some time the linear approximation fails to be valid and the nonlinearities should be taken into account. By analogy with bifurcation theory for low dimensional dynamical systems, one may wonder if there exists some kind of normal form for extended systems. This is indeed the case and the corresponding equation is often called an amplitude equation (for reasons that should become clear shortly). We will first derive the amplitude equation for the Swift-Hohenberg evolution by a multi-scale analysis which albeit non rigorous provides many insights about the problem. This method is Physically very natural and has been made rigorous in the context of averaging. As we have seen above, there are several scales in the problem. We have first the space scale unity which is obvious from the coefficients of the equation being of order one (in particular the coefficients in front of the partial derivatives). We have also seen that this is the typical scale of the periodic structure. We have also seen above the occurrence of a new large scale of order ϵ^{-1}. In the discussion of the instability we have also seen emerging a time scale of order $\epsilon^{-2} = (\mu - \mu_c)^{-1}$. The idea is to put all these scales together in a sort of functional model for the solution, namely to look for a function of three variables and of the parameter $u(\tau, x_1, x_2, \epsilon)$ such that $u(\epsilon^2 t, x, \epsilon x, \epsilon)$ would be a solution of (14). Moreover we will search for this solution as a power series expansion in the parameter ϵ, namely

$$u(\epsilon^2 t, x, \epsilon x, \epsilon) = u_0(\epsilon^2 t, x, \epsilon x) + \epsilon u_1(\epsilon^2 t, x, \epsilon x) + \epsilon^2 u_2(\epsilon^2 t, x, \epsilon x) + \text{h.o.t.}$$

Although the occurrence of the two correlated variables x and ϵx may look strange from a mathematical point of view, it makes sense physically if we think of very different scales evolving independently. In any case, we can plug this ansatz into the equation and try to solve it order by order in ϵ. The reader can check that the main effect is the replacement of the differential operator ∂_x by the sum $\partial_{x_1} + \epsilon \partial_{x_2}$, and of the operator ∂_t by the operator $\epsilon^2 \partial_\tau$. Collecting the zeroth order terms we get

$$0 = -\big(1 + \partial_{x_1}^2\big)^2 u_0 - u_0^3 \ .$$

This is a fourth order ordinary differential equation, whose only bounded solution is $u_0 = 0$ (see the next exercise).

Exercise 4.11. Prove that if u_0 is a bounded solution of the above equation and φ is the function (2), then

$$\int \varphi[u_0^2 + 2u_0 u_0'' + u_0''^2]\, dx + 2 \int \varphi'' u_0 u_0''\, dx - \frac{1}{2} \int \varphi^{(4)} u_0^2\, dx + \int \varphi u_0^4\, dx = 0 \ .$$

Deduce that there is a constant $C > 0$ such that for any b (the parameter in φ) and any bounded solution u_0 of the above equation we have

$$\int \varphi u_0^4\, dx \leq C b^2 \big\| u_0 \big\|_{L^\infty}^2 \ .$$

Dividing by b and letting b tend to zero conclude that $u_0 = 0$.

We now come to the terms of order one in ϵ. We get since $u_0 = 0$

$$0 = -\big(1 + \partial_{x_1}^2\big)^2 u_1 \ .$$

This equation has a basis of four complex solutions, but since we want real solutions, we have

$$u_1(\tau, x_1, x_2) = A(\tau, x_2) e^{ix_1} + B(\tau, x_2) x_1 e^{ix_1} + \text{c.c.}$$

where c.c. denotes the complex conjugate. Since we want bounded solutions, we set $B = 0$ (these are the analogs of the secular terms coming from resonances in mechanics). To order two in ϵ we obtain

$$0 = -\big(1 + \partial_{x_1}^2\big)^2 u_2 - 4\big(1 + \partial_{x_1}^2\big) \partial_{x_1} \partial_{x_2} u_1 \ .$$

Note however that from the above expression for u_1, the last term is zero. We therefore get for u_2 the (bounded) solution

$$u_1(\tau, x_1, x_2) = C(\tau, x_2)e^{ix_1} + \text{c.c.}$$

We now come to the more interesting third order. We get (after canceling some terms)

$$\partial_\tau u_1 = -\big(1 + \partial_{x_1}^2\big)^2 u_3 - 4\partial_{x_1}^2 \partial_{x_2}^2 u_1 + u_1 - u_1^3 \,.$$

It is interesting to rewrite this equation in the following form using the above expression for u_1

$$\big(1 + \partial_{x_1}^2\big)^2 u_3 = -e^{ix_1}\left\{\partial_\tau A - 4\partial_{x_2}^2 A - A + 3A|A|^3\right\} - e^{3ix_1}A^3 + \text{c.c.}$$

We now observe that e^{ix} is in the kernel of the (self adjoint) operator $\big(1 + \partial_{x_1}^2\big)^2$. therefore in order to obtain a bounded solution in x_1, the coefficient should vanish. We obtain the equation for A

$$\partial_\tau A = 4\partial_{x_2}^2 A + A - 3A|A|^2$$

which except for a trivial change of scales is the Ginzburg-Landau equation (1) with parameters $\alpha = \beta = 0$. This is the amplitude equation we were looking for, the analog of the normal form for low dimensional dynamical systems. We therefore expect that the solutions of the Swift-Hohenberg equation (14) behave for $\mu = \epsilon^2$ small as

$$u(t, x) = \frac{\epsilon}{\sqrt{3}} e^{ix} A(\epsilon^2 t, \epsilon x/2) + \text{c.c.}$$

where the function A is a solution of the Ginzburg-Landau equation (1). In particular we see the physical nature of the solution: a structure of wavelength of order one (the function ϵe^{ix}) with a slow space-time modulation of its amplitude A.

As mentioned above this kind of argument can be made rigorous in various ways, and we now formulate a theorem.

Theorem 4.1. *There exists positive constants* $c_1, \ldots, c_4$ *and a positive number* ϵ_0 *such that for any* $u_0 \in C_{b,u}^2$*, and any* $\epsilon \in]0, \epsilon_0[$*, there exists a positive number* $T = T(u_0, \epsilon)$ *such that for any* $t > T$*, the solution* u *of (14) satisfies* $\|u(t, \cdot)\| \le c_1\epsilon$*. Moreover, there is a function* $A(\tau, y)$ *solution of the Ginzburg-Landau equation*

$$\partial_\tau A = \partial_y^2 A + A - A|A|^2$$

such that if $0 \le s \le c_2\epsilon^{-2}\log\epsilon^{-1}$*, we have*

$$\|u(t + s, \cdot,) - v(s, \cdot,)\|_{L^\infty} \le c_3\epsilon^{1+c_4}$$

where

$$v(s, x) = \frac{\epsilon}{\sqrt{3}} e^{ix} A(\epsilon^2 t, \epsilon x/2) + \text{c.c.}$$

In other words, modulo an error of higher order we have constructed the solutions of the Swift-Hohenberg equation from those of the Ginzburg-Landau equation. Many similar computations for other extended systems lead to the Ginzburg-Landau equation as an amplitude equation, this is similar to the generic normal forms of low dimensional dynamical systems. Although for extended systems it would be hard to develop a generic theory, the renormalisation group approach of the next subsection explains why the Ginzburg-Landau equation is universal (in the sense of renormalisation group). It is easy to verify that the equation

$$\partial_\tau A = \partial_y^2 A - A|A|^2$$

is invariant by simultaneous scaling of space, time and amplitude. Namely, if $A(t, x)$ is a solution, then for any $\lambda > 0$, the function $\lambda A(\lambda^2 t, \lambda x)$ is also a solution. In some sense this equation can be considered as a fixed point of the renormalisation. Although we will not pursue the analysis at this general level, the reader familiar with the renormalisation group argument in statistical mechanics will recognise these ideas behind the developments of the next subsection. We refer to [2] [6] for the general approach.

4.3 Renormalisation.

The renormalisation group approach is well suited to the study of large space and large time behaviour of various systems. For the application to partial differential equations we refer to [2] [6] [32] [4].

The basic idea is roughly as follows. Start by fixing a time scaling factor $S > 1$. Consider the map describing the time evolution from time S^n to time S^{n+1}, i.e. which from the initial data at time S^n integrate the equation over the time interval $[S^n, S^{n+1}]$ leading in particular to the value of the solution at time S^{n+1}. Of course, in most cases one cannot construct this map explicitly but one can get many information about it (see section 2). Doing the same thing on the time interval $[S^{n+1}, S^{n+2}]$ one obtains another map. By scaling space, time (by a factor S^{-1}) and the solution, and eventually performing more complex transformations, can one compare these two maps? (in the classical renormalisation group dogma, suitable scalings and transformations should lead to a converging sequence of maps).

To implement this program here, it is convenient to start by fixing a space scaling factor $\Lambda > 1$. The value of this factor is irrelevant (provided it is larger than one), one may take for definiteness $\Lambda = 2$. This leads to a sequence of (large) space scales (Λ^n). We can now associate a sequence of (small) scales $\big(\delta_n\big)$ for the wave number by

$$\delta_n = \Lambda^{-n} \ .$$

As we have seen above, this leads to a sequence of (large) time scales $\big(\delta_n^{-2},$

diffusive scaling), in other words $S = \lambda^2$. Also from the above analysis, $\big(\delta_n\big)$ should also serve as a sequence of (small) scales for the solutions.

We will also need a sequence of small scales for the unstable windows of wave numbers. From figure 2 this should be $\big(\delta_n\big)$. It turns out from the technical details below that this is slightly too narrow, and we will instead correct these scales by a sequence of (moderately large) factors $\big(\theta_n\big)$. It is convenient to choose

$$\theta_n = \big(\delta_n^{-1}\big)^{\zeta}$$

where ζ is a small positive number, for example $\zeta = 1/5$ will work.

Although we have used the previous analysis to relate some of the scales, this is not a-priori necessary. One can start with general (exponential) sequences of scales, perform the computations below, and one will discover that the only case where a non trivial result is found is when the scales are related as above.

We have described above what should be a renormalisation group step: integrate the equation on a time interval, then rescale and eventually perform some more complicated transformations. This leads to the idea that all renormalisation steps are in fact identical. So we start with a scale δ for the solution (which should be though as δ_n for some n), namely we will assume from now on that the (real) initial condition u_0 satisfies

$$\Lambda^{-1}\delta \leq \|u_0\|_{L^\infty} \leq \delta .$$

One should think of δ as a small number but which may be of order one. The lower bound here is for the optimal definition of δ but is not necessary in the following arguments.

We also have a relative scale θ for a window of wave numbers, namely $\theta = \delta^{-\zeta}$. We now associate to this scale three operators which in some sense cut the support of the Fourier transform. In the space $L^2(dx)$ these will be projections obtained by multiplying the Fourier transform by the characteristic function of an interval. Since the Fourier transforms of the functions we are interested in are distributions, we cannot cut them sharply, and we will use multiplications by smoother functions. The operators will not be projections but this is not too important. We fix once for all a cut-off function ψ which is infinitely differentiable, non negative, with support in $[-2, 2]$ and which is equal to one on the interval $[-1, 1]$. We then define a function $K_{\delta,\theta}$ by

$$K_{\delta,\theta}(x) = \delta\theta\big(\mathcal{F}\psi\big)(\delta\theta x)$$

where $\mathcal{F}$ denotes the Fourier transform. Convolution by $K_{\delta,\theta}$ localises the support of the Fourier transform to the interval $[-2\delta\theta, 2\delta\theta]$.

Exercise 4.12. Prove it.

We now define three operators $P^+_{\delta,\theta}$, $P^-_{\delta,\theta}$, and $R_{\delta,\theta}$ by their kernels

$$P^+_{\delta,\theta}(y, x) = e^{i(y-x)}K_{\delta,\theta}(y - x) \ ,$$

$$P^-_{\delta,\theta}(y,x) = e^{-i(y-x)} K_{\delta,\theta}(y-x) \ ,$$

and

$$R_{\delta,\theta} = \mathrm{Id} - P^+_{\delta,\theta} - P^-_{\delta,\theta} \ .$$

From the above remarks, we see that $P^+_{\delta,\theta}$ localises the support of the Fourier transform near $+1$ and $P^-_{\delta,\theta}$ localises the support of the Fourier transform near -1.

Exercise 4.13. Prove it.

We now formulate some useful results about these operators. Define the constant C_1 by

$$C_1 = 2\left(1 + \sup_{0\le j\le 5} \left\|\partial_x^j \mathcal{F}\psi\right\|_{L^1}\right) \ . \tag{19}$$

Exercise 4.14. Prove that this constant is finite.

Lemma 4.1. *For any $\theta > 1 > \delta > 0$ we have*

1. *the operators $P^+_{\delta,\theta}$, $P^-_{\delta,\theta}$ and $R_{\delta,\theta}$ are bounded in L^∞ with norm smaller than C_1.*
2. *If $u \in L^\infty$ is real, then $P^-_{\delta,\theta}u = \overline{P^+_{\delta,\theta}u}$.*
3. *The operator $P^+_{\delta,\theta}$ maps L^∞ in $C^4_{b,u}$, and more precisely for any $u \in L^\infty$ we have*
$$\sup_{0\le j\le 4} \left\|(\delta\theta)^{-j}\partial_x^j\big(P^+_{\delta,\theta}u\big)\right\|_{L^\infty} \le C_1\|u\|_{L^\infty} \ .$$
   The same estimates hold for $P^-_{\delta,\theta}$.

Exercise 4.15. Prove this lemma.

We now denote by A the solution of the Ginzburg-Landau equation

$$\partial_t A = A'' + \eta\delta^{-2}A - A|A|^3 \tag{20}$$

with initial condition

$$A(0,x) = \sqrt{3}\delta^{-1}\big(P^+_{\delta,\theta}u_0\big)(2\delta^{-1}x) \ . \tag{21}$$

We will denote by $\mathcal{A}$ the function of space and time

$$\mathcal{A}(t,x) = \frac{1}{\sqrt{3}} e^{ix} A(\delta^2 t, \delta x/2) + \text{c.c.} \tag{22}$$

We define the remainder r by

$$r(t,x) = u(t,x) - \delta\mathcal{A}(t,x) \ . \tag{23}$$

It is easy to verify that this function satisfies the following equation

$$\partial_t r = -(1+\partial_x^2)^2 r + \eta r - 3\delta^2\mathcal{A}^2 r - 3\delta\mathcal{A}r^2 - r^3 + \delta\mathcal{R}\ , \tag{24}$$

where

$$\mathcal{R} = -(1+\partial_x^2)^2\mathcal{A} + \eta\mathcal{A} - \delta^2\mathcal{A}^3 - \partial_t\mathcal{A} \tag{25}$$

and with initial condition

$$r(0,\,\cdot\,) = R_{\delta,\theta}u_0\ .$$

Exercise 4.16. Verify these assertions.

With these notations one can establish the following result which is the technical basis of the renormalisation step.

Lemma 4.2. *There are constants* $C_6 > 1$ *and* $C_7 > 1$ *such that*

1. *If* $1 > \delta^2 > \eta$*, if* $t < C_6\delta^{-2}$ *and* $\|r(0,\,\cdot\,)\|_{L^\infty} < (1+2C_1)\delta$*, then*
$$\|r((t,\,\cdot\,)\|_{L^\infty} < C_7\delta\ .$$
2. *There are positive constants* $C_8 > 1$ *and* $C_9 > 1$ *such that under the above constraints on* η*,* δ *and* $r(0,\,\cdot\,)$*, we have for* $C_6\delta^{-2} \le t \le C_8\delta^{-2}\log\delta^{-1}$
$$\|r((t,\,\cdot\,)\|_{L^\infty} < C_9\delta\left(e^{-t\delta^2\theta^2/C_9} + \theta^{-1}e^{C_9 t\delta^2}\right)\ .$$

We postpone the proof of this lemma to the end of this section.

We will also need a control over the time evolution of A.

Lemma 4.3. *There is a constants* $C_2 > C_1$ *(the constant of equation (19)) independent of* δ *and* θ *such that for any* $\delta\theta^4 < 1$ *and* $\delta > \eta^{1/2}$*, if* A *evolves according to equation (20) with initial condition* $A(0,\,\cdot\,)$ *satisfying*

$$\sup_{0\le j\le 4}\theta^{-j}\|\partial_x^j A(0,\,\cdot\,)\|_{L^\infty} \le C_1\ ,$$

and $\mathcal{R}$ *is defined by equation (25), then for any* $t \ge 0$ *we have*

$$\sup_{0\le j\le 4}\theta^{-j}\|\partial_x^j A(t,\,\cdot\,)\|_{L^\infty} \le C_2\ ,$$

$$\|\mathcal{R}(t,\,\cdot\,) + \mathcal{C}(t,\,\cdot\,)\|_{L^\infty} \le C_2\delta^3\theta^4\ ,$$

and

$$\|\mathcal{R}(t,\,\cdot\,)\| \le C_2\delta^2\ .$$

where

$$\mathcal{C}(t,x) = \frac{1}{3^{3/2}}\delta^2 e^{3ix}A^3(\delta^2 t, \delta x\sqrt{2}) + \text{c.c.}$$

Moreover, there exists $c_1 > 0$ *and* $t_0 > 0$ *independent of* δ*,* θ*,* η *and* A *such that if* $\delta > c_1\eta^{1/2}$*, for any* $t > t_0$ *we have*

$$\|A(t,\,\cdot\,)\|_{L^\infty} \le \frac{1}{4\Lambda}\ .$$

Exercise 4.17. Prove this lemma by changing the scales and using the results of section 2 (in particular Lemma 2.2 and estimations of the derivatives). Observe in particular the cancellations leading to the second estimate.

After these preparatory definition and Lemmas we can now give the recursive proof of Theorem 4.1. We start by selecting an integer n_0 large enough such that $\log \delta_{n_0}^{-1} > 1 + (\Lambda + 1)^2$, and

$$C_9 \left(e^{-C_8 \theta_{n_0}^2 \log \delta_{n_0}^{-1} / C_9} + \theta_{n_0}^{-1} e^{C_9 \log \delta_{n_0}^{-1}} \right) \leq \frac{1}{2\Lambda} \ .$$

Note that these constraints on n_0 are independent of η. We define $\eta_0 = \Lambda^{-4} \delta_{n_0}^2 / 8$.

Using a-priori estimate as in section one, one can prove that for any $u_0 \in C_{b,u}^0$, there is a time $t(u_0) \geq 0$ such that

$$\big\| u(t(u_0), \,\cdot\,) \big\|_{L^\infty} \leq \delta_{n_0}$$

Since the proof is recursive, we assume we have reached a stage corresponding to the scale δ_n ($n \geq n_0$), in other words we assume that u_0 satisfies

$$\big\| u_0 \big\|_{L^\infty} \leq \delta_n \ .$$

We will also assume that

$$\Lambda^{-1} \delta_n \leq \big\| u_0 \big\|_{L^\infty} \tag{26}$$

which says that for the given u_0 our choice of n is optimal. This is not necessary for the proof but not requiring this condition will result in worse estimates.

We now deal with the first part of the proof and assume for the moment that $\delta_n^2 > \eta$. We consider the time evolution starting with the initial condition u_0 and will apply Lemma 4.2 with $\delta = \delta_n$ and $\theta = \theta_n = \delta_n^{-\zeta}$.

We first construct the function A by solving the Ginzburg-Landau equation (20) with initial condition (21). With this function we construct the function $\mathcal{A}$ using the definition (22). We also have the solution u of the Swift-Hohenberg equation (14) and we can define the remainder r using equation (23) which evolves according to equation (24).

It is convenient to split the time evolution into two parts. We define a first time t_n by

$$t_n = \left[C_4 \delta_n^{-2} \right] \ .$$

On the time interval $[0, t_n]$ we apply the first part of Lemma 4.2 and obtain

$$\sup_{0 \leq t \leq t_n} \big\| r(t, \,\cdot\,) \big\|_{L^\infty} \leq C_5 \delta_n \ .$$

Indeed, from our choice of δ_n it follows from Lemma 4.1 that

$$\big\| r(0, \,\cdot\,) \big\|_{L^\infty} \leq (1 + 2C_1) \delta_n \ .$$

Note that during this time interval, the bound on the size of the remainder does not improve.

We now define a second time $T_n > t_n$ by

$$T_n = [C_8 \delta_n^2 \log \delta_n^{-1}] \ ,$$

where C_8 is the constant appearing in Lemma 4.2.

On the time interval $[t_n, T_n]$ we apply the second part of Lemma 4.2. This implies that there is a time $\mathcal{T}_n \in [t_n, T_n]$ such that

$$\big\| r(\mathcal{T}_n, \,\cdot\,) \big\|_{L^\infty} \le \frac{\delta_n}{2\Lambda} \ .$$

Using Lemma 4.3 we also have

$$\big\| 3^{-1/2} A(\delta_n^2 \mathcal{T}_n, \delta_n \,\cdot\,) \big\|_{L^\infty} \le \frac{1}{4\Lambda} \ .$$

Combining these two estimates we get using equations (22) and (23)

$$\big\| u(\mathcal{T}_n, \,\cdot\,) \big\|_{L^\infty} \le \delta_{n+1} \ .$$

We can now repeat the same argument at level $n+1$ starting with $u_0 = u(\mathcal{T}_n, \,\cdot\,)$. This proves recursively the first part of Theorem 4.1.

The second part is obtained by applying only one of the above step with $\delta_n \simeq \eta$, discarding condition (26). The result follows immediately from both parts of Lemma (4.2). This finishes the proof of Theorem 4.1.

Remark 4.1. After a time of order $\mathcal{O}(1)\eta^{-1} \log \eta^{-1}$, the errors amplify and the Theorem does not apply anymore. This is similar to what happens for normal forms in low dimensional dynamical systems. It was however observed by Eckhaus [16] that at the time when the errors start to grow, one can apply Theorem 4.1 with a new initial condition for the Ginzburg-Landau equation, and one will be able to accurately imitate the solution for the Swift-Hohenberg equation again for an interval of time of order $\mathcal{O}(1)\eta^{-1} \log \eta^{-1}$. Proceeding like this one gets an infinite sequence of intervals where the solution of equation (14) is shadowed by functions constructed from solutions of the Ginzburg-Landau equation.

4.4 Proof of Lemma 4.2.

We now give the proof of Lemma 4.2, which requires first some preparatory results.

We will also need some results on the free linear evolution. We denote by L the differential operator given by

$$Lv = -\big(1 + \partial_x^2\big)^2 v + \eta v \ .$$

Lemma 4.4. *There exists a positive constant* C_3 *such that*

1. *for any* $0 < \eta < 1$ *and any* $t > 0$ *we have*

$$\left\| e^{tL} \right\|_{L^\infty} \le C_3 e^{\eta t} \ .$$

2. *For any* $0 < \eta < 1$*, any* $t > 0$*, and for any* $\delta^2\theta^2 < 1$ *we have*

$$\left\| e^{tL} R_{\delta,\theta} \right\|_{L^\infty} \le C_3 e^{t(\eta - \delta^2\theta^2/8)} \ .$$

Proof. Since L is a differential operator with constant coefficients, the operator e^{tL} is a convolution operator with the function $e^{\eta t} I(t,x)$ where

$$I(t,x) = \int e^{-t(1-k^2)^2} e^{ikx} dk \ .$$

We leave as an exercise to the reader to check the following estimates. There is a constant $C > 0$ such that for any $0 < t < 1$ and any x we have

$$\left| I(t,x) \right| \le C \min\left\{ t^{-1/4}, \frac{t^{1/4}}{x^2} \right\} \ .$$

This estimate follows from the observation that if p is an integer we have

$$x^p I(t,x) = (-i)^p \int e^{-t(1-k^2)^2} \partial_k^p e^{ikx} dk \ .$$

One can then integrate by parts p times over k and estimate the integral, observing to simplify the estimation that for $0 \le t \le 1$ one has

$$e^{-t(1-k^2)^2} \le \mathcal{O}(1) e^{-tk^4/2} \ .$$

For $t > 1$ the estimate is somewhat similar. We first split I into a sum of two integrals, one for $k > 0$ and one for $k < 0$. The estimations of these two integrals are similar. We have after a simple change of variables

$$\int_0^\infty e^{-t(1-k^2)^2} e^{ikx} dk = e^{ix} \int_{-1}^\infty e^{-tu^2(1+u)^2} e^{iux} du \ .$$

The integral is estimated as above multiplying by some power of x and integrating by parts. The estimation can be further simplified by observing that for $u > -1$

$$e^{-tu^2(2+u)^2} \le e^{-tu^2} \ .$$

One concludes that For any $t \ge 1$ and any x we have

$$\left| I(t,x) \right| \le C \min\left\{ t^{-1/2}, \frac{t^{1/2}}{x^2} \right\} \ .$$

This leads immediately to the existence of a constant $C' > 0$ such that for any $t > 0$

$$\int |I(x,t)|\, dx \le C' \ .$$

The first statement of the lemma follows immediately from Young's inequality. The second statement of the lemma is obtained in a similar way. We have to estimate the integral

$$\int (1 - \psi((1-k)/\delta\theta) + \psi((1+k)/\delta\theta)) e^{-t(1-k^2)^2} e^{ikx} dk \ .$$

In particular, the integrations will be limited to the outside of the intervals $[-1-\delta\theta, -1+\delta\theta]$ and $[1-\delta\theta, 1+\delta\theta]$. We leave as an exercise to the reader to check that the announced estimate holds (distinguish the cases $t < 1/(\delta\theta)^2$ and $t \ge 1/(\delta\theta)^2$). □

Remark 4.2. In the first statement of the Lemma we have a global estimate which is somewhat expected and does not improve in time. On the contrary, in the second part we see an interesting contraction appearing which will be the basis of the convergence argument.

We will need also an estimate on the growth rate of the solution of the linear equation

$$\partial_t v = Lv - 3\delta^2 \mathcal{A}^2 v - 3\delta \mathcal{A} r v - r^2 v \ , \tag{27}$$

(which for $v = r$ is equation (24)). This equation is non autonomous since the functions $\mathcal{A}$ and r may depend on time. We will denote by $U(t, t_0)$ for $t > t_0$ the linear operator mapping the initial condition at time t_0 to the solution at time t.

Lemma 4.5. *For the operators $U(t, t_0)$ we have the following estimates.*

1. *There is a constant $C_4 > 1$ such that for any $0 < t_0 < t$ and any $0 < \eta < 1/2$ we have*

   $$\big\| U(t,t_0) \big\|_{L^\infty} \le C_4 e^{C_4 (t-t_0)(\eta + \gamma(t,t_0))} \ ,$$

   where

   $$\gamma(t,t_0) = \sup_{t_0 \le s \le t} (\delta^2 \|\mathcal{A}(s,\,\cdot\,)\|_{L^\infty}^2 + \|r(s,\,\cdot\,)\|_{L^\infty}^2) \ .$$

2. *There is a constant $C_5 > 1$ such that for any $\eta < 1/2$, for any $\delta^2 > \eta$, for any $\delta\theta < 1$ and for any $t > t_0 > 0$ we have*

   $$\big\| U(t,t_0) R_{\delta,\theta} \big\|_{L^\infty} \le C_5 e^{-(t-t_0)\delta^2\theta^2/C_5}$$

   $$+ C_5 \theta^{-2} \delta^{-2} (\delta^2 + \gamma(t,t_0)) e^{C_5 (t-t_0)(\delta^2 + \gamma(t,t_0))} e^{C_5 (t-t_0)\delta^2} \ .$$

Proof. To prove the first part we rewrite equation (27) as an integral equation

$$v(t,\,\cdot\,) = e^{(t-t_0)L}v(t_0,\,\cdot\,) - \int_0^{t-t_0} e^{sL}\left(3\delta^2\mathcal{A}^2 v + 3\delta\mathcal{A}rv + r^2 v\right)(t-s,\,\cdot\,)\,ds\;. \tag{28}$$

It follows that

$$\big\|v(t,\,\cdot\,)\big\|_{L^\infty} \le \Big\|e^{(t-t_0)L}\Big\|_{L^\infty}\big\|v(t_0,\,\cdot\,)\big\|_{L^\infty}$$

$$+\int_0^{t-t_0}\big\|e^{sL}\big\|_{L^\infty}\big\|\left(3\delta^2\mathcal{A}^2 v + 3\delta\mathcal{A}rv + r^2 v\right)(t-s,\,\cdot\,)\big\|_{L^\infty}\,ds\;.$$

Using the first part of Lemma 4.4 we get

$$\big\|v(t,\,\cdot\,)\big\|_{L^\infty} \le C_3 e^{\eta t}\big\|v(t_0,\,\cdot\,)\big\|_{L^\infty}$$

$$+\int_0^{t-t_0} C_3 e^{\eta s}\big\|\left(3\delta^2\mathcal{A}^2 v + 3\delta\mathcal{A}rv + r^2 v\right)(t-s,\,\cdot\,)\big\|_{L^\infty}\,ds\;.$$

The result now follows at once from Gronwall Lemma (see [22]).

We now give a proof of the second part of the Lemma. For $t-t_0<1$ this follows at once from the first part. For $t-t_0\ge 1$, we write $t-t_0=n+\tau$ with $n\in\mathbf{N}$ and $0\le\tau<1$. For $s>s_0$ we define the operator $\mathcal{M}(s,s_0)$ by

$$\mathcal{M}(s,s_0) = U(s,s_0) - e^{(s-s_0)L}\;.$$

Note that it follows from equation (28) that

$$\mathcal{M}(s,s_0) = \int_0^{s-s_0} e^{\sigma L} M_{s-\sigma} U(s-\sigma,s_0)\,d\sigma$$

where M_s is the operator of multiplication by the function

$$-\left(3\delta^2\mathcal{A}^2 + 3\delta\mathcal{A}r + r^2\right)(s,\,\cdot\,)\;.$$

It follows immediately from Lemma 4.3 and the first part of the present Lemma that there is a constant $\tilde{C}>1$ such that for any $s_0>0$ we have

$$\sup_{s_0\le s\le s_0+1}\big\|\mathcal{M}(s,s_0)\big\|_{L^\infty} \le \tilde{C}(\delta^2+\gamma(t,t_0))\,e^{\tilde{C}(\delta^2+\gamma(t,t_0))} \tag{29}$$

We now observe that

$$U(t,t_0)R_{\delta,\theta} = U(t,t_0+n)\cdots U(t_0+2,t_0+1)U(t_0+1,t_0)R_{\delta,\theta}$$

$$= e^{(t-t_0)L}R_{\delta,\theta} + \mathcal{M}(t,t_0+n)e^{nL}R_{\delta,\theta}$$

$$+\sum_{j=0}^{n-1} U(t,t_0+j+1)\mathcal{M}(t_0+j+1,t_0+j)e^{jL}R_{\delta,\theta}\;.$$

Using Lemma 4.4, estimate (29) and the first part of the present Lemma we get

$$\|U(t,t_0)R_{\delta,\theta}\|_{L^\infty}$$
$$\leq C_3 e^{(t-t_0)(\eta-\delta^2\theta^2/8)} + C_3\tilde{C}(\delta^2+\gamma(t,t_0)) \times\ e^{\tilde{C}(\delta^2+\gamma(t,t_0))} e^{n(\eta-\delta^2\theta^2/8)}$$
$$+C_3\tilde{C}(\delta^2+\gamma(t,t_0))\, e^{\tilde{C}(\delta^2+\gamma(t,t_0))} \sum_{j=0}^{n-1} e^{C_4(n-j)(\delta^2+\gamma(t,t_0))} e^{j(\eta-\delta^2\theta^2/8)}\ .$$

The result follows at once by estimating the geometric sum over j. ☐

Finally we can now give a proof of Lemma 4.2. Let σ be the non decreasing function defined by

$$\sigma(t) = \delta^{-1} \sup_{0\leq s\leq t} \|r(s,\,\cdot\,)\|_{L^\infty}\ .$$

It follows immediately from equation (24) that

$$r(t,\,\cdot\,) = U(t,0)r(0,\,\cdot\,) + \delta\int_0^t U(t,t-s)\mathcal{R}(s,\,\cdot\,)\,ds\ .$$

Note that the unknown function r also occurs in the definition of U, but not in the definition of $\mathcal{R}$. From Lemma 4.3 and the definition of γ in Lemma 4.5 we have

$$\gamma(t,0) \leq \delta^2(C_1^2+\sigma(t)^2)\ .$$

Using Lemma 4.5 we get

$$\sigma_t \leq C_4\sigma_0 e^{C_4 t(\eta+\delta^2(C_1^2+\sigma(t)^2))} + \delta^2\int_0^t C_4C_2 e^{sC_4(\eta+\delta^2(C_1^2+\sigma(s)^2))} ds\ .$$

Denote by t_* the smallest positive number such that $\sigma_{t_*} \geq C_7 = C_4(1+2C_1)$. It follows immediately that $t_* \geq \mathscr{O}(1)\delta^{-2}$.

This proves the first part of Lemma 4.2. To prove the second part, we observe that since $P^{\pm}_{\delta,\theta} r(0,\,\cdot\,) = 0$ we have

$$r(t,\,\cdot\,) = U(t,0)R_{\delta,\theta}r(0,\,\cdot\,) + \delta\int_0^t U(t,t-s)\mathcal{R}(s,\,\cdot\,)\,ds$$
$$= U(t,0)R_{\delta,\theta}r(0,\,\cdot\,) + \delta\int_0^t U(t,t-s)\big(\mathcal{R}(s,\,\cdot\,)+\mathcal{C}(s,\,\cdot\,)\big)\,ds$$
$$-\delta\int_0^t U(t,t-s)P^{+}_{\delta,\theta}\,\mathcal{C}(s,\,\cdot\,)\,ds - \delta\int_0^t U(t,t-s)P^{-}_{\delta,\theta}\,\mathcal{C}(s,\,\cdot\,)\,ds$$
$$-\delta\int_0^t U(t,t-s)R_{\delta,\theta}\,\mathcal{C}(s,\,\cdot\,)\,ds$$

Lemma 4.3 asserts that $\mathcal{R} + \mathcal{C}$ is small and we now have to control

$$\mathcal{C}(t,x) = \frac{1}{3^{3/2}}\delta^2 e^{3ix} A^3(\delta^2 t, \delta x\sqrt{2}) + \text{c.c.}\ .$$

The idea is that if A would be constant, then the Fourier transform of $\mathcal{C}$ would have no support in the unstable zone (near $k = \pm 1$). Therefore we can expect $P^{\pm}_{\delta,\theta}\mathcal{C}$ to be small. We have

$$P^{+}_{\delta,\theta}\mathcal{C}(t,x) = \int e^{i(x-y)} K_{\delta,\theta}(x-y)\,\mathcal{C}(t,y)dy$$

$$= \frac{1}{3^{3/2}}\delta^2 e^{ix}\int e^{2iy} K_{\delta,\theta}(x-y) A^3(\delta^2 t, \delta y\sqrt{2})dy$$

$$+\frac{1}{3^{3/2}}\delta^2 e^{ix}\int e^{-4iy} K_{\delta,\theta}(x-y)\overline{A}^3(\delta^2 t, \delta y\sqrt{2})dy\ .$$

We now explain how to estimate the first integral, the second one being treated similarly. Observing that

$$e^{2iy} = -\frac{i}{2}\frac{d}{dy}e^{2iy}$$

we can integrate by parts and get

$$\int e^{2iy} K_{\delta,\theta}(x-y) A^3(\delta^2 t, \delta y\sqrt{2})dy = -\frac{i}{2}\int e^{2iy} K'_{\delta,\theta}(x-y) A^3(\delta^2 t, \delta y\sqrt{2})dy$$

$$+\frac{3i\delta}{\sqrt{2}}\int e^{2iy} K_{\delta,\theta}(x-y) A^2(\delta^2 t, \delta y\sqrt{2}) A'(\delta^2 t, \delta y\sqrt{2})dy\ .$$

Using the definition of $K_{\delta,\theta}$ and Lemma 4.3, we get easily

$$\left\| P^{+}_{\delta,\theta}\mathcal{C}(t,\,\cdot\,)\right\|_{L^\infty} \le c\delta^3\theta$$

where c is a positive constant independent of t, δ, θ and A.

Using the second part of Lemma 4.5 we obtain

$$\sigma(t) \le \Big(C_5 e^{-t\delta^2\theta^2/C_5} + C_5\theta^{-2}(C_1^2 + \sigma(t)^2)e^{C_5 t\delta^2(C_1^2+\sigma(t)^2)}\Big)\, e^{\eta t}\big(\sigma(0) + t\delta^2 C_1^3\big)$$

$$+tC_4 e^{(1+C_4)t(\eta+\delta^2(C_1^2+\sigma(t)^2)}(2c\delta^3\theta + C_2\delta^4\theta^3)\ .$$

This is where it is convenient to choose for θ^{-1} a small power of δ. The second part of Lemma 4.2 follows immediately.

5 Dimension and Entropy.

In order to simplify the discussion we will work with the Ginzburg-Landau equation in space dimension d

$$\partial_t A = (1+i\alpha)\Delta A + A - (1+i\beta)A|A|^2 \ , \tag{30}$$

although the results are easy to extend to many other equations.

We first recall a result in bounded domains. Consider equation (30) on a d dimensional torus of size $L > 0$, namely on $[0, L]^d$ with periodic boundary conditions. Denote by $\mathscr{A}_L$ the globally attracting set (in C^2 for example).

For this system, Ghidaglia and Heron ([20]) proved that there are three constants $c_1 > c_2 > 0$ and $L_0 > 0$ (depending only on α, β and the dimension d) such that for any $L > L_0$

$$c_2 L^d \le \dim\bigl(\mathscr{A}_L\bigr) \le c_1 L^d \ .$$

Here dim is the box counting dimension which is defined as follows. For $\epsilon > 0$, and a precompact subset $\mathscr{B}$ of $\mathbf{R}^d$, let $N_\epsilon(\mathscr{B})$ be the smallest number of balls of radius ϵ needed to cover $\mathscr{B}$. The box counting dimension is defined by

$$\dim(\mathscr{B}) = \limsup_{\epsilon \searrow 0} \frac{\log N_\epsilon(\mathscr{B})}{\log \epsilon^{-1}} \ . \tag{31}$$

We mention that Ghidaglia and Heron obtained similar results for the Hausdorff dimension and also for other boundary conditions.

If we now consider the Ginzburg-Landau equation in $\mathbf{R}^d$ it is easy to verify that $\mathscr{A}_L \subset \mathscr{A}$. By this we mean that functions on the torus can be identified with periodic functions in $\mathbf{R}^d$. From the Ghidaglia-Heron result we immediately obtain $\dim(\mathscr{A}) = \infty$. Thinking in terms of extensive quantities like in statistical mechanics, it would be better to write the Ghidaglia-Heron bound as

$$c_2 \le \frac{\dim\bigl(\mathscr{A}_L\bigr)}{L^d} \le c_1 \ .$$

It is then natural to ask if the quantity $L^{-d}\dim\bigl(\mathscr{A}_L\bigr)$ has a limit when L tends to infinity.

If one interprets the dimension as the number of degrees of freedom excited in the system, this would say that the number of degrees of freedom is proportional to the volume of the domain. Unfortunately, up to now this is still an open question.

In order to progress in this direction, one can go back to the idea that the attractor in unbounded domain is observed only in bounded windows (eventually through a sequence of windows of growing size). This idea was mentioned by Shannon at the end of his famous paper on the entropy (see [30]) and later developed by Kolmogorov and Tikhomirov (see [25]). To simplify the notations I will from now on assume that the space dimension is $d = 1$ although the arguments extend to higher dimension (sometimes with more technical work required).

In dimension one, a window on the infinite line is simply an interval I. The globally attracting set $\mathscr{A}$ defined as in section II for some topology (for example $H^1_{\varphi,u}$) observed in I will be denoted by $\mathscr{A}_{|I}$. We recall (warn the reader) that this is not the attractor in I but the restriction to I of all the functions in $\mathscr{A}$. We also adopt a topology for the functions on I which is weaker than the topology used to define $\mathscr{A}$. For example, we will start with the topology of $L^\infty(I)$. Since $\mathscr{A}$ is composed of functions analytic and bounded in a strip around the real axis, the set $\mathscr{A}_{|I}$ is precompact in $L^\infty(I)$. At this point we still have a problem since $\dim_{L^\infty}\big(\mathscr{A}\big) = \infty$ (the notation $\dim_{L^\infty}$ emphasises that we are using L^∞ balls to cover the set). This result comes from the fact that if a function is analytic in a strip around the real axis, knowing that function on an interval is enough to know it everywhere.

The idea of Shannon Kolmogorov and Tikhomirov called the ϵ-entropy circumvents this difficulty by exchanging the order of the limits which have to be taken. For $\epsilon > 0$, let $N_\epsilon(I)$ be the smallest number of balls in $L^\infty(I)$ needed to cover $\mathscr{A}_{|I}$. We then have the easy but important Lemma.

Lemma 5.1. *If I and J are two closed intervals with disjoint interior we have*

$$N_\epsilon(I \cup J) \le N_\epsilon(I) N_\epsilon(J)\ .$$

Proof. Let $B_1, \ldots, B_{N_\epsilon(I)}$ be $N_\epsilon(I)$ balls of radius ϵ in $L^\infty(I)$ covering $\mathscr{A}_{|I}$, namely

$$\mathscr{A}_{|I} \subset \bigcup_{j=1}^{N_\epsilon(I)} B_j\ .$$

We will denote by $f_j \in L^\infty(I)$ the center of the ball B_j.

Similarly, let $C_1, \ldots, C_{N_\epsilon(J)}$ be $N_\epsilon(J)$ balls of radius ϵ in $L^\infty(J)$ covering $\mathscr{A}_{|J}$, namely

$$\mathscr{A}_{|J} \subset \bigcup_{j=1}^{N_\epsilon(J)} C_j\ .$$

We will denote by $g_j \in L^\infty(J)$ the centerer of the ball C_j. For any $1 \le r \le N_\epsilon(I)$, and any $1 \le s \le N_\epsilon(J)$ we define the function

$$h_{r,s} = \chi_I f_r + \chi_J g_s\ .$$

It is easy to verify that this function belongs to $L^\infty(I \cup J)$. We denote by $D_{r,s}$ the ball of radius ϵ in $L^\infty(I \cup J)$ centered on $h_{r,s}$. It is easy to see that

$$\mathscr{A}_{|I\cup J} \subset \bigcup_{r,s} D_{r,s}\ .$$

This implies form the definition of $N_\epsilon(I \cup J)$ that

$$N_\epsilon(I \cup J) \leq N_\epsilon(I) N_\epsilon(J) \ .$$

$\square$

We now recall a well known lemma on sub-additive functions.

Lemma 5.2. *Let a be a function from $\mathbf{R}^+$ to $\mathbf{R}$ satisfying for any $x, y \geq 1$ the inequality $a(x+y) \leq a(x) + a(y)$. Then*

$$\lim_{x \to \infty} \frac{a(x)}{x} = \inf_x \frac{a(x)}{x} \ ,$$

and in particular, the limit exists. If a is bounded below, the limit is finite.

Exercise 5.1. Prove this lemma.

Applying this result to the function $a(L) = \log N_\epsilon([-L, L]) \geq 0$ and using translation invariance, we immediately obtain the following Theorem.

Theorem 5.1. *For any $\epsilon > 0$ the following limit exists and is finite*

$$\mathcal{S}_\epsilon^{L^\infty}(\mathscr{A}) = \lim_{L \to \infty} \frac{\log N_\epsilon([-L, L])}{2L}$$

it is called the ϵ-entropy per unit length.

Proof. By translation invariance, we have for any $L > 0$ and $L' > 0$

$$N_\epsilon([-L-L', L+L']) = N_\epsilon([0, 2L+2L']) \ .$$

Using Lemma 5.1 and translation invariance again we get

$$N_\epsilon([0, 2L+2L']) \leq N_\epsilon([0, 2L]) N_\epsilon([2L, 2L+2L']) = N_\epsilon([-L, L]) N_\epsilon([-L', L']) \ .$$

The result follows immediately from Lemma 5.2. $\square$

One would like to apply this idea also for the other spaces of section II like $C^0_{b,u}$, $C^2_{b,u}$, $H^2_{\varphi,u}$ etc. However an interesting approximation problem appears when concatenating balls from disjoint intervals. Consider two closed intervals I and J with disjoint interior but a common boundary point as illustrated in figure 3 (beware of the optical illusion in this picture).

In each interval I, J we have a ball of radius ϵ in C^0 which is represented by its center: the dotted graph, and its two boundaries: the dashed graphs. Any continuous function in this ball has a graph between the two dashed lines. We also have represented (the solid graph) a continuous function on $I \cup J$ which belongs to the two balls. In L^∞ we constructed a ball for $I \cup J$ by simply abutting the two centers. The main point is that we obtained a ball in $L^\infty(I \cup J)$ with the same diameter ϵ. If we do this here we obtain

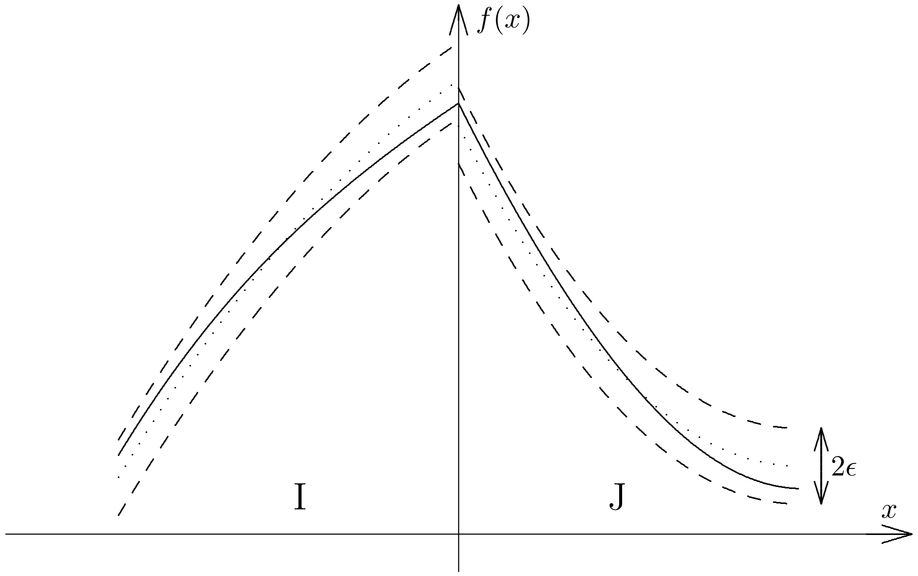

Figure 3: Two abutting balls

a discontinuous function (which of course still belongs to L^∞). In other words in spaces of continuous (regular) functions we need a more subtle construction when passing from coverings in I and J to coverings in $I \cup J$. In dimension one this is given by the following theorem. Note also that if we relax the condition of having the same diameter ϵ for the ball in $I \cup J$, the problem becomes easy but this is useless for our purpose.

Theorem 5.2. *Let I and J be two closed intervals of length larger than one with disjoint interiors and a common boundary point. Let $\mathcal{F}$ be a bounded subset of $C^s(I \cup J)$, $s \geq 0$. Then there is an integer k which depends only on s and the diameter of $\mathcal{F}$ but not on the intervals I and J such that for any $0 < \epsilon \leq 1$, if B_I and C_J are balls in $C^s(I)$ and $C^s(J)$ respectively of radius ϵ and such that $B_I \cap C_J \cap \mathcal{F} \neq \emptyset$, then there exists at most k balls $D_1, \ldots, D_k$ in $C^s(I \cup J)$ with the same radius ϵ such that*

$$B_I \cap C_J \cap \mathcal{F} \subset \bigcup_{j=1}^{k} D_j \ .$$

We refer to [13] for a proof of this result. As far as I know there is no published proof in the case of space dimension larger than one. One may conjecture that in space dimension $d > 1$ the number k should be of the order of the length of the common boundary (surface in dimension 3 etc.).

From this result we derive at once that

$$N_\epsilon^{C^1}(I \cup J) \leq k N_\epsilon^{C^1}(I) N_\epsilon^{C^1}(J) \ .$$

Where $N_\epsilon^{C^1}(I)$ denotes the minimum number of balls of radius ϵ in C^1 needed to cover $\mathscr{A}_{|I}$.

Exercise 5.2. Let a be a function from $\mathbf{R}^+$ to $\mathbf{R}$ satisfying for some constant c and for any $x, y \geq 1$ the inequality $a(x+y) \leq a(x) + a(y) + c$. Then $\lim_{x\to\infty} a(x)/x$ exists. If a is bounded below, the limit is finite.

We conclude immediately that the ϵ-entropy per unit length exists in $C^1_{b,u}$, namely

$$\mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A}) = \lim_{L\to\infty} \frac{\log N_\epsilon^{C^1_{b,u}}([-L,L])}{2L} \ .$$

The behaviour in ϵ of $\mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A})$ is also known to some extent.

Theorem 5.3. *There exists two constants $C_2 > C_1 > 0$ such that for any $0 < \epsilon < 1/2$ we have*

$$C_1 \log \epsilon^{-1} \leq \mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A}) \leq C_2 \log \epsilon^{-1} \ .$$

The same result holds in other topologies like $C^r_{b,u}$, moreover for any r we have

$$\lim_{\epsilon\to 0} \frac{\mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A})}{\mathcal{S}_\epsilon^{L^\infty}(\mathscr{A})} = 1 \ .$$

We refer to [13] and [11] for a proof. See also [18].

Remark 5.1. It is not known whether the quantity $\mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A})/\log\epsilon^{-1}$ has a limit when ϵ tends to zero.

Remark 5.2. Assume that $D(\mathscr{A}) = \lim_{\epsilon\to 0} \mathcal{S}_\epsilon^{C^1_{b,u}}(\mathscr{A})/\log\epsilon^{-1}$ exists. The heuristics of the above Theorem is that

$$N_\epsilon^{C^1_{b,u}}([-L,L]) \approx \epsilon^{-2LD(\mathscr{A})} \epsilon^{-\mathcal{O}(1)\log\epsilon^{-1}}$$

where the last term can be guessed from the details of the proof. We see the difference when taking the limits in different orders. If we take the logarithm, divide by L and take the limit $L \to \infty$, the second term does not contribute. On the other hand, if we fix L, take the logarithm, divide by $\log\epsilon^{-1}$ and let ϵ tend to zero we get an infinite result. This is consistent with the fact that the attractor has infinite dimension (see formula (31)).

In their original work, Kholmogorov and Tikhomirov used the ϵ-entropy per unit length to measure the relative size of function spaces. More precisely, for any $h > 0$ and any $C > 0$ let

$$S_h(C) = \left\{ f \text{ analytic in } |\Im z| \leq h \text{ and satisfying } \sup_{z,\,|\Im z|\leq h} |f(z)| \leq C \right\} \ .$$

It is proved in [25] that for any $h > 0$ and any $C > 0$

$$\lim_{\epsilon \to 0} \frac{\mathcal{S}_\epsilon^{L^\infty}\big(S_h(C)\big)}{(\log \epsilon^{-1})^2} = \frac{1}{\pi h} \ .$$

Note that it follows from the results of section 2 that $\mathscr{A} \subset S_h(C)$ for some h and C. However Theorem 5.3 shows that the ϵ-entropy per unit length is much smaller, namely there are far less functions in $\mathscr{A}$ than in $S_h(C)$. This is because we have for $\mathcal{S}_\epsilon^{L^\infty}\big(\mathscr{A}\big)$ an estimate of the order $\log \epsilon^{-1}$ instead of $(\log \epsilon^{-1})^2$. In other words

$$\lim_{\epsilon \searrow 0} \frac{\mathcal{S}_\epsilon^{L^\infty}\big(\mathscr{A}\big)}{\mathcal{S}_\epsilon^{L^\infty}\big(S_h(C)\big)} = 0 \ .$$

On the other hand, for any $\sigma > 0$ and $C > 0$ let

$$E_\sigma(C) = \Big\{ f \text{ entire satisfying for any } z \in \mathbf{C} \text{ the bound } |f(z)| \le C e^{\sigma |\Im z|} \Big\} \ .$$

It is proved in [25] that for any $\sigma > 0$ and any $C > 0$

$$\lim_{\epsilon \to 0} \frac{\mathcal{S}_\epsilon^{L^\infty}\big(E_\sigma(C)\big)}{\log \epsilon^{-1}} = \frac{2\sigma}{\pi} \ .$$

In other words, if we define the dimension per unit length D by

$$D(\,\cdot\,) = \lim_{\epsilon \to 0} \frac{\mathcal{S}_\epsilon^{C^1}(\,\cdot\,)}{\log \epsilon^{-1}} \ ,$$

and taking the lim sup if the limit does not exits, we have $D(S_h(C)) = \infty$, $D(E_\sigma(C)) = 2\sigma/\pi$ and $D(\mathscr{A}) < \infty$. However $\mathscr{A} \neq E_\sigma(C)$.

Exercise 5.3. Show that the function $u(x) = \tanh(x/\sqrt{2})$ is a stationary solution of the Ginzburg-Landau equation (30) with $\alpha = \beta = 0$. Show that this implies $\mathscr{A} \neq E_\sigma(C)$.

Similar results can be established for the topological entropy per unit length. We refer to [11] for details and references.

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