Existence, Regularity, Stability and Boundedness For Some Partial Functional Differential Equations [1]

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Abstract: The purpose of this work is to discuss the fundamental theory for a class of partial functional differential equations with a non dense domain. We assume that the linear part is not necessarily densely defined and satisfies the known Hille-Yosida condition. We suppose that the nonlinear part is locally Lipschitz continuous. We prove the existence of mild and strict solutions. When the solutions are globally defined, we give a linearized principal near an equilibrium. We investigated the stability and the asymptotic behavior of solutions of the linear equations. We prove that the behavior of solution is completely obtained by the so-called characteristic equation. We study the existence of bounded solutions for nonhomogeneous equation. We prove the existence of periodic or almost periodic solutions when the solution semigroup of the linear equation is hyperbolic. Finally, we propose to study the stability for one dimensional reaction diffusion model with delay arising in physical systems.

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1. **Introduction**

By a partial functional differential equation, we mean an evolution system described by the following differential equation,

$$\left\{\begin{array}{l} \dfrac{du}{dt}(t)=Au(t)+F(u_t), \quad t\geq 0 \\ u_0=\varphi\in\mathcal{C}_E, \end{array}\right. \tag{1.1}$$

where $\mathcal{C}_E := \mathcal{C}([-r,0]\,;E)$, $r>0$, denotes the space of continuous functions from $[-r,0]$ to a Banach space E with the uniform convergence topology and $A : D(A)\subseteq E\to E$ is a linear operator. For $u\in\mathcal{C}([-r,b]\,;E)$, $b>0$ and $t\in[0,b]$, u_t denotes, as usual, the element of $\mathcal{C}_E$ defined by

$$u_t(\theta)=u(t+\theta), \text{ for } \theta\in[-r,0]\,.$$

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The following model provides an example of such situation,

(1.2)
$$\begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + \displaystyle\int_{-r}^{0} w(t+\theta,\xi)d\mu(\theta) + \varrho(t,x), \text{ for } t \geq 0 \text{ and } 0 \leq \xi \leq \pi, \\ w(t,0) = w(t,\pi) = 0, \text{ for } t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi), \text{ for } -r \leq \theta \leq 0 \text{ and } 0 \leq \xi \leq \pi, \end{cases}$$

where the initial data u_0 is a given function from $[-r,0] \times [0,\pi]$ to $\mathbb{R}$, μ is a positive measure on $[-r,0]$, $\varrho : \mathbb{R}\times [0,\pi] \to \mathbb{R}$ is a continuous function. Several classes of differential equations, such as delayed reaction diffusion equations, wave equations and age-dependent population equations can be reformulated as partial functional differential equations. For more details about this topics, we refer to the book [47], which provides several examples to demonstrate the fact that partial functional differential equations arise from various physical systems including population ecology, control theory, climate models, structured population models,.....

It is well known that if A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ in E or, equivalently

(**i**) $\overline{D(A)} = E$,

(**ii**) there exist $M \geq 0$, $\omega \in I\!R$ such that if $\lambda > \omega$ $(\lambda I - A)^{-1} \in \mathcal{L}(E)$ and

$$\left|(\lambda-\omega)^n (\lambda I - A)^{-n}\right| \leq M, \text{ for all } n \in \mathbb{N},$$

then, the classical semigroup theory ensures the well posedness of Problem (1.1). See for example Travis and Webb [44], Webb [45], Fitzgibbon [30], Kunish and Schappacher [36] and the references therein. In [44], [45], [30] and [36], the authors assumed that F is globally Lipschitz continuous from $\mathcal{C}_E$ to E. They proved there results by using the following variation-of-constants formula

$$u(t,\varphi) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s)F(u_s(.,\varphi))ds, \text{ for } t \geq 0 \\ \varphi(t), \text{ for } t \in [-r,0]. \end{cases}$$

More recently, Parrott [40] established a result of local existence for Eq.(1.1), in the case where A satisfies **(i)** and **(ii)** and F satisfies a locally Lipschitz condition. She used, in the same paper, a result of Desch and Schappacher [20] to develop a principle of linearized stability for Eq.(1.1).

In [2] and [28], Eq.(1.1) has been studied in the case when A has a non-dense domain and (**ii**) holds. But, in this case, the perturbation considered is globally Lipschitz continuous.

In the present work we are investigating the case when : A satisfies (**ii**) with a non-dense domain; and, F satisfies a locally Lipschitz condition. We prove the local existence of solutions of Eq.(1.1) and in the case when F is globally Lipschitz continuous, we study the problem of linearized stability near an equilibrium point. The method of proofs is similar to that of Webb [46] and Parrott [39], [40].

Let us end this introduction with some considerations about the usefulness of such results. Why to bother working with operators defined in non-dense domain ?

First of all, there are many examples in the main domains where evolution equations can be not densely defined. One can refer for this to [19] who provide many examples. Non-density occurs, in many situations, from restrictions made on the space where the equation is considered or from boundary conditions (for example, the space $\mathcal{C}^1$ with null value on the boundary is non-dense in the space of continuous functions). All these operators are naturally associated with delay differential equations in a variety of applications: population dynamics is an example. As a trivial example, one can think of the classical Lotka-Von Foerster equation, considered in the continuous case, and where one would assume that individuals die with probability 1 beyond a certain age. One could object that by suitably restriction the problem could be reduced to a C_0-semigroup. This does not work here since there is no reason in general that the function F takes its values in the closure of the domain, that is to say, in the space where A determines a C_0-semigroup. On the other hand, the integrated semigroups theory allows the range of the operator F to be in a subset of E.

2. **Integrated semigroups**

In this section, we collect some known definitions and results.

Definition 2.1. [12] Let E be a Banach space. A family $(S(t))_{t\geq 0}\subset \mathcal{L}(E)$ is called an integrated semigroup if the following conditions are satisfied :
(i) $S(0)=0$,
(ii) for any $x\in E$, $S(t)x$ is a continuous function of $t\geq 0$ with values in E,
(iii) for any $t,s\geq 0 \quad S(s)S(t)=\displaystyle\int_0^s (S(t+\tau)-S(\tau))d\tau.$

Definition 2.2. [12] An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $M\geq 0$ and $\omega\in I\!R$ such that

$$|S(t)|\leq Me^{\omega t}\quad \text{for } t\geq 0.$$

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda):=\lambda\displaystyle\int_0^{+\infty} e^{-\lambda t}S(t)dt$ exists for all $\lambda>\omega$.
We have the following general definition.

Definition 2.3. [12] An operator A is called a generator of an integrated semigroup, if there exists $\omega\in I\!R$ such that $(\omega,+\infty)\subset\rho(A)$ (the resolvent set of A), and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0)=0$ and $R(\lambda,A)=\lambda\displaystyle\int_0^{+\infty} e^{-\lambda t}S(t)dt$ for all $\lambda>\omega$, where $R(\lambda,A):=(\lambda I-A)^{-1}$, for $\lambda\in\rho(A)$.

Similar results as for semigroups can be obtained for integrated semigroups.

Proposition 2.4. [12] *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x\in E$ and $t\geq 0$,*

$$\int_0^t S(s)xds\in D(A) \text{ and } S(t)x=A\left(\int_0^t S(s)xds\right)+tx.$$

Moreover, for all $x\in D(A),\ t\geq 0$

$$S(t)x\in D(A),\qquad\qquad AS(t)x=S(t)Ax$$

and

$$S(t)x = \int_0^t S(s)Ax\,ds + tx.$$

Corollary 2.5. [12] *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$ one has $S(t)x \in \overline{D(A)}$. Moreover, let $x \in E$. Then $S(\cdot)x$ is right-sided differentiable in $t \geq 0$ if and only if $S(t)x \in D(A)$. In that case*

$$S'(t)x = AS(t)x + x.$$

An important special case is when the integrated semigroup is locally Lipschitz continuous with respect to time.

Definition 2.6. [37] An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous, if for all $\tau > 0$ there exists a constant $k(\tau) > 0$ such that

$$|S(t) - S(s)| \leq k(\tau)\,|t - s|\,, \qquad \text{for all } t, s \in [0, \tau]\,.$$

In this case, we know from [37], that $(S(t))_{t\geq 0}$ is exponentially bounded.

Definition 2.7. [37] We say that a linear operator A is a Hille-Yosida operator if there exist $M \geq 0$ and $\omega \in I\!R$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega)^n \left|R(\lambda, A)^n\right|,\, n \in \mathbb{N},\, \lambda > \omega\right\} \leq M.$$

The following theorem shows that the Hille-Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 2.8. [37] *The following assertions are equivalent.*
(**i**) *A is the generator of a locally Lipschitz continuous integrated semigroup,*
(**ii**) *A is a Hille-Yosida operator.*

Remark 1. If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ on E, then, the part A_F of A in $F = \overline{D(A)}$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on F and we have, for $x \in F$, $S(t)x = \int_0^t T(s)(x)ds$, $t \geq 0$. However, for $x \in E \setminus F$ the function $t \to S(t)x$ is not differentiable at any $t \geq 0$.

In the sequel, we give some results for the existence of solutions of the following Cauchy problem

$$\left\{\begin{array}{l} \dfrac{du}{dt}(t) = Au(t) + f(t), \quad t \geq 0, \\ u(0) = x \in E, \end{array}\right. \tag{2.1}$$

where A is a Hille-Yosida operator on E, without being densely defined. By a solution of Eq.(2.1) on $[0, T]$ where $T > 0$, we understand a function $u \in \mathcal{C}^1\left([0, T]\,, E\right)$ satisfying $u(t) \in D(A)$ for $t \in [0, T]\,$, such that the two relations in (2.1) hold.
The following definition is due to Da Prato and Sinestrari.

Definition 2.9. [19] Given $f \in L^1_{loc}(0, +\infty; E)$ and $x \in E$, we say that $u : [0, +\infty) \to E$ is an integral solution of Eq.(2.1) if the following assertions are true
(**i**) $u \in \mathcal{C}([0, +\infty)\,; E)$,
(**ii**) $\int_0^t u(s)ds \in D(A)$, for $t \geq 0$,
(**iii**) $u(t) = x + A\left(\int_0^t u(s)ds\right) + \int_0^t f(s)ds$, for $t \geq 0$.

Remark 2. From this definition, we deduce that for an integral solution u, we have $u(t) \in \overline{D(A)}$, for all $t > 0$, because $u(t) = \lim\limits_{h \to 0} \frac{1}{h} \int_t^{t+h} u(s)ds$ and $\int_t^{t+h} u(s)ds \in D(A)$. In particular, $x \in \overline{D(A)}$ is a necessary condition for the existence of an integral solution of Eq.(2.1). This is suggestive to solve Eq.(2.1) by the variation-of-constants formula where $S(t)$ is the integrated semigroup generated by A.

Theorem 2.10. [18] *Suppose that A is a Hille-Yosida operator on E, $x \in \overline{D(A)}$ and $f : [0, +\infty) \to E$ is a continuous function. Then Eq.(2.1) has a unique integral solution which is given by the following variation of constants formula*

$$u(t) = S'(t)x + \frac{d}{dt} \int_0^t S(t-s)f(s)ds, \textit{for } t \geq 0, \tag{2.2}$$

Furthermore, the function u satisfies the following estimate,

$$|u(t)| \leq Me^{\omega t}(|x| + \int_0^t e^{-\omega s} |f(s)|\, ds), \ \textit{for } t \geq 0.$$

Note that Theorem 2.10 also says that $\int_0^t S(t-s)f(s)ds$ is differentiable with respect to t.

3. **Local existence and regularity of solutions**

In the following, we assume that

($\mathbf{H}_1$) A is a Hille-Yosida operator.

($\mathbf{H}_2$) $F : \mathcal{C}_E \to E$ is locally Lipschitz continuous, i.e., for each $\rho > 0$ there exists a constant $C_0(\rho) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{C}_E$ and $|\varphi_1|, |\varphi_2| \leq \rho$ then

$$|F(\varphi_1) - F(\varphi_2)| \leq C_0(\rho) |\varphi_1 - \varphi_2|.$$

We know from Theorem 2.8 that A is the generator of a locally Lipschitz continuous integrated semigroup $(S(t))_{t \geq 0}$ on E and $|S(t)| \leq Me^{\omega t}$, for $t \geq 0$.

Definition 3.1. We say that a function $u : [-r, +\infty) \to E$ is an integral solution of Eq.(1.1) if the following conditions are true,
(i) $u \in \mathcal{C}([-r, +\infty); E)$,
(ii) $u_0 = \varphi$,
(iii) $\int_0^t u(s)ds \in D(A)$, for $t \geq 0$,
(iv) $u(t) = \varphi(0) + A\left(\int_0^t u(s)ds\right) + \int_0^t F(u_s)ds$, for $t \geq 0$.

Definition 3.2. We say that a function $u : [-r, +\infty) \to E$ is a strict solution of Eq.(1.1) if the following conditions hold,
(i) $u \in \mathcal{C}^1([0, +\infty); E)$ or $\mathcal{C}([0, T]; D(A))$
(ii) $u_0 = \varphi$,
(iii) u satsfies Eq.(1.1), for $t \geq 0$,

From the closedness property of the operator A, we can see that the following result hold.

Proposition 3.3. (**i**) *If u is an integral solution of Eq.(1.1) in $[-r,a]$, then for all $t \in [0,a]$, $u(t) \in \overline{D(A)}$. In particular $\varphi(0) \in \overline{D(A)}$.*
(**ii**) *If u is an integral solution of Eq.(1.1) in $[-r,a]$, such that u belongs to $C^1([0,a];E)$ or $C([0,a];D(A))$, then u is also a strict solution of Eq.(1.1) in $[-r,a]$.*

By theorem 2.10, if the integral solution u exists, then it is given by the following variation of constants formula,

$$u(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(u_s)ds, \text{ for } t \in [0,T[\\ \varphi(t), \text{for } t \in [-r,0]. \end{cases} \tag{3.1}$$

Theorem 3.4. *Suppose that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied. Let $\varphi \in \mathcal{C}_E$ be such that $\varphi(0) \in \overline{D(A)}$. Then, there exists a maximal interval of existence $[-r,T_\varphi[$, $T_\varphi > 0$, and a unique integral solution $u(.,\varphi)$ of Eq.(1.1), defined on $[-r,T_\varphi[$ and either*

$$T_\varphi = +\infty \quad \text{or} \quad \limsup_{t\to T_\varphi^-} |u(t,\varphi)| = +\infty.$$

Moreover, $u(t,\varphi)$ is a continuous function of φ, in the sense that if $\varphi \in \mathcal{C}_E$, $\varphi(0) \in \overline{D(A)}$ and $t \in [0,T_\varphi[$, then there exist positive constants L and ε such that, for $\psi \in \mathcal{C}_E$, $\psi(0) \in \overline{D(A)}$ and $|\varphi - \psi| < \varepsilon$, we have

$$t \in [0,T_\psi[\text{ and } |u(s,\varphi) - u(s,\psi)| \le L|\varphi - \psi|, \text{ for all } s \in [-r,t].$$

Proof. Note that $(\mathbf{H}_2)$ implies that, for each $\rho > 0$ there exists $C_0(\rho) > 0$ such that for $\varphi \in \mathcal{C}_E$ and $|\varphi| \le \rho$, we have

$$|F(\varphi)| \le C_0(\rho)|\varphi| + |F(0)| \le \rho C_0(\rho) + |F(0)|.$$

Let $\varphi \in \mathcal{C}_E$, $\varphi(0) \in \overline{D(A)}$, $\rho = |\varphi| + 1$, $c_1 = \rho C_0(\rho) + |F(0)|$ and $T_1 > 0$. Consider the following set

$$Z_\varphi = \left\{ u \in \mathcal{C}([-r,T_1];E) : u(s) = \varphi(s) \text{ if } s \in [-r,0] \text{ and } \sup_{0\le s\le T_1} |u(s) - \varphi(0)| \le 1 \right\},$$

where $\mathcal{C}([-r,T_1],E)$ is endowed with the uniform convergence topology. It's clear that Z_φ is a closed set of $\mathcal{C}([-r,T_1],E)$. Consider the mapping

$$H : Z_\varphi \to \mathcal{C}([-r,T_1],E)$$

which is defined by

$$H(u)(t) = \begin{cases} S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(u_s)ds, \text{ for } t \in [0,T_1] \\ \varphi(t), \text{ for } t \in [-r,0]. \end{cases}$$

Note that a fixed point of H is an integral solution of Eq.(1.1).We will show that

$$H(Z_\varphi) \subseteq Z_\varphi.$$

Let $u \in Z_\varphi$ and $t \in [0,T_1]$, we have, for suitable constants M and ω

$$\begin{aligned} |H(u)(t) - \varphi(0)| &\le |S'(t)\varphi(0) - \varphi(0)| + \left| \frac{d}{dt}\int_0^t S(t-s)F(u_s)ds \right|, \\ &\le |S'(t)\varphi(0) - \varphi(0)| + Me^{\omega t}\int_0^t e^{-\omega s}|F(u_s)|ds. \end{aligned}$$

We can assume, without loss of generality, that $\omega > 0$. Then,

$$|H\left(u\right)(t)-\varphi(0)| \leq |S'(t)\varphi(0)-\varphi(0)| + Me^{\omega t}\int_0^t |F(u_s)|\,ds.$$

Since $|u(s)-\varphi(0)| \leq 1$, for $s \in [0,T_1]$, and $\rho = |\varphi|+1$, we obtain $|u(s)| \leq \rho$, for $s \in [-r,T_1]$. Then, $|u_s| \leq \rho$, for $s \in [0,T_1]$ and

$$\begin{aligned} |F(u_s)| &\leq C_0(\rho)\,|u_s| + |F(0)|, \\ &\leq c_1. \end{aligned}$$

Consider the constant $T_1 > 0$ sufficiently small such that

$$\sup_{0\leq s\leq T_1}\{|S'(s)\varphi(0)-\varphi(0)| + Me^{\omega s}c_1 s\} < 1.$$

We deduce that

$$\begin{aligned} |H\left(u\right)(t)-\varphi(0)| &\leq |S'(t)\varphi(0)-\varphi(0)| + Me^{\omega t}c_1 t \\ &< 1. \end{aligned}$$

Hence,

$$H\left(Z_\varphi\right) \subseteq Z_\varphi.$$

On the other hand, let $u, v \in Z_\varphi$ and $t \in [0,T_1]$. We have

$$\begin{aligned} |H\left(u\right)(t)-H\left(v\right)(t)| &= \left|\frac{d}{dt}\int_0^t S(t-s)(F(u_s)-F(v_s))ds\right|, \\ &\leq Me^{\omega t}\int_0^t |F(u_s)-F(v_s)|\,ds, \\ &\leq Me^{\omega t}C_0\left(\rho\right)\int_0^t |u_s-v_s|\,ds, \\ &\leq Me^{\omega T_1}C_0\left(\rho\right)T_1\,|u-v|_{\mathcal{C}([-r,T_1],E)}\,. \end{aligned}$$

Note that $\rho \geq 1$, then

$$\begin{aligned} Me^{\omega T_1}C_0(\rho)T_1 &\leq Me^{\omega T_1}c_1T_1, \\ &\leq \sup_{0\leq s\leq T_1}\{|S'(s)\varphi(0)-\varphi(0)| + Me^{\omega s}c_1 s\}, \\ &< 1. \end{aligned}$$

Then, H is a strict contraction in Z_φ and H has one and only one fixed point u in Z_φ. We conclude that Eq.(1.1) has one and only one integral solution which is defined on the interval $[-r,T_1]$. Let $u(.,\varphi)$ be the unique integral solution of Eq.(1.1), defined on it's maximal interval of existence $[0,T_\varphi[$, $T_\varphi > 0$. Assume that $T_\varphi < +\infty$ and

$$\limsup_{t\to T_\varphi^-}|u(t,\varphi)| < +\infty.$$

Then, there exists a constant $\rho > 0$ such that $|u(t,\varphi)| \leq \rho$, for $t \in [-r,T_\varphi[$. Let $t, t+h \in [0,T_\varphi[$, $h > 0$, and $\theta \in [-r,0]$. If $t+\theta \geq 0$, we obtain

$$\begin{aligned} &|u(t+\theta+h,\varphi)-u(t+\theta,\varphi)| \leq |(S'(t+\theta+h)-S'(t+\theta))\varphi(0)| \\ &\qquad + \left|\frac{d}{dt}\int_0^{t+\theta+h} S(t+\theta+h-s)F(u_s,\varphi)ds - \frac{d}{dt}\int_0^{t+\theta} S(t+\theta-s)F(u_s,\varphi)ds\right|, \end{aligned}$$

$$\leq |S'(t+\theta)| \, |S'(h)\varphi(0) - \varphi(0)| + \left| \frac{d}{dt} \int_{t+\theta}^{t+\theta+h} S(s) F(u_{t+\theta+h-s}, \varphi) ds \right| + \left| \frac{d}{dt} \int_0^{t+\theta} S(s) \left(F(u_{t+\theta+h-s}, \varphi) - F(u_{t+\theta-s}, \varphi) \right) ds \right|.$$

This implies that,

$$\begin{aligned} |u_{t+h}(\theta, \varphi) - u_t(\theta, \varphi)| \; & \leq M e^{\omega T_\varphi} |S'(h)\varphi(0) - \varphi(0)| + M e^{\omega T_\varphi} c_1 h \\ & + M e^{\omega T_\varphi} C_0(\rho) \int_0^t |u_{s+h}(., \varphi) - u_s(., \varphi)| \, ds. \end{aligned}$$

If $t + \theta < 0$. Let $h_0 > 0$ sufficiently small such that, for $h \in]0, h_0[$

$$|u_{t+h}(\theta, \varphi) - u_t(\theta, \varphi)| \; \leq \sup_{-r \leq \sigma \leq 0} |u(\sigma + h, \varphi) - u(\sigma, \varphi)|.$$

Consequently, for $t, t+h \in [0, T_\varphi[\,, \; h \in]0, h_0[\,;$

$$\begin{aligned} |u_{t+h}(., \varphi) - u_t(, \varphi)| \; & \leq \delta(h) + M e^{\omega T_\varphi} \left(|S'(h)\varphi(0) - \varphi(0)| + c_1 h \right) \\ & + M e^{\omega T_\varphi} C_0(\rho) \int_0^t |u_{s+h}(., \varphi) - u_s(., \varphi)| \, ds, \end{aligned}$$

where

$$\delta(h) = \sup_{-r \leq \sigma \leq 0} |u(\sigma + h, \varphi) - u(\sigma, \varphi)|.$$

By Gronwall's Lemma, it follows

$$|u_{t+h}(., \varphi) - u_t(, \varphi)| \leq \beta(h) \exp\left[C_0(\rho) M e^{\omega T_\varphi} T_\varphi \right],$$

with

$$\beta(h) = \delta(h) + M e^{\omega T_\varphi} \left[|S'(h)\varphi(0) - \varphi(0)| + c_1 h \right].$$

Using the same reasoning, one can show the same result for $h < 0$. It follows immediately, that $\lim_{t \to T_\varphi^-} u(t, \varphi)$ exists. Consequently, $u(., \varphi)$ can be extended to T_φ, which contradicts the maximality of $[0, T_\varphi[\,$.

Next, we prove now that the solution depends continuously on the initial data. Let $\varphi \in \mathcal{C}_E$, $\varphi(0) \in \overline{D(A)}$ and $t \in [0, T_\varphi[$. We put

$$\rho = 1 + \sup_{-r \leq s \leq t} |u(s, \varphi)|$$

and

$$c(t) = M e^{\omega t} \exp\left(M e^{\omega t} C_0(\rho) t \right).$$

Let $\varepsilon \in]0, 1[$ such that $c(t)\varepsilon < 1$ and $\psi \in \mathcal{C}_E$, $\psi(0) \in \overline{D(A)}$ such that

$$|\varphi - \psi| < \varepsilon.$$

We have

$$|\psi| \leq |\varphi| + \varepsilon < \rho.$$

Let

$$T_0 = \sup \left\{ s > 0 : |u_\sigma(., \psi)| \leq \rho \text{ for } \sigma \in [0, s] \right\}.$$

If we suppose that $T_0 < t$, we obtain for $s \in [0, T_0]$,

$$|u_s(., \varphi) - u_s(., \psi)| \leq M e^{\omega t} |\varphi - \psi| + M e^{\omega t} C_0(\rho) \int_0^s |u_\sigma(., \varphi) - u_\sigma(., \psi)| \, d\sigma.$$

By Gronwall's Lemma, we deduce that

$$|u_s(,\varphi) - u_s(,\psi)| \leq c(t)\,|\varphi - \psi|\,. \tag{3.2}$$

This implies that

$$|u_s(.,\psi)| \leq c(t)\varepsilon + \rho - 1 < \rho, \text{ for all } s \in [0, T_0]\,.$$

It follows that T_0 cannot be the largest number $s > 0$ such that $|u_\sigma(.,\psi)| \leq \rho$, for $\sigma \in [0, s]$. Thus, $T_0 \geq t$ and $t < T_\psi$. Furthermore, $|u_s(.,\psi)| \leq \rho$, for $s \in [0, t]$, then using the inequality (3.2) we deduce the dependence continuous with the initial data.

Theorem 3.5. *Assume that the hypothesis of Theorem 5.2 hold. Furthermore, assume that $F : \mathcal{C}_E \to E$ is continuously differentiable and $F' : \mathcal{C}_E \to \mathcal{L}(\mathcal{C}_E, E)$ satisfies the locally Lipschitz condition $(\mathbf{H}_2)$, i.e., for each $\rho > 0$ there exists a constant $C_1(\rho) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{C}_E$ and $|\varphi_1|, |\varphi_2| \leq \rho$ then*

$$|F'(\varphi_1) - F'(\varphi_2)| \leq C_1(\rho)\,|\varphi_1 - \varphi_2|\,.$$

Let for given $\varphi \in \mathcal{C}^1_E := \mathcal{C}^1([-\tau, 0], E)$ such that

$$\varphi(0) \in D(A), \ \ \varphi'(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A\varphi(0) + F(\varphi),$$

and $u(.,\varphi) : [-r, T_\varphi[\to E$ be the unique integral solution of Eq.(1.1). Then, $u(.,\varphi)$ is a strict solution of Eq.(1.1) on $[-r, T_\varphi[$.

Proof. Let $\varphi \in \mathcal{C}^1_E$ such that $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$. Let $u := u(.,\varphi)$ be the unique integral solution of Eq.(1.1) on $[-r, T_\varphi[$ and $T_1 \in]0, T_\varphi[$. It's clear that, there exists a unique function $v : [0, T_1] \to E$ which solves the following integral equation

$$v(t) = \begin{cases} S'(t)\varphi'(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F'(u_s)v_s ds, \text{ for } t \in [0, T_1]\,, \\ \varphi'(t), \text{for } t \in [-r, 0]\,. \end{cases}$$

Define the function w by

$$w(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t v(s)ds, \text{ for } t \in [0, T_1] \\ \varphi(t), \text{ for } t \in [-r, 0]\,. \end{cases}$$

We will prove that $u = w$. Using the expression satisfied by v, we obtain, for $t \in [0, T_1]$

$$w(t) = \varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)F'(u_s)v_s ds.$$

We have $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + F(\varphi)$, then

$$S(t)\varphi'(0) = S(t)A\varphi(0) + S(t)F(\varphi).$$

Using Corollary 2.5, we deduce that

$$S(t)\varphi'(0) = S'(t)\varphi(0) - \varphi(0) + S(t)F(\varphi).$$

Moreover, we have

$$\int_0^t S(t-s)\,F(w_s)\,ds = \int_0^t S(s)\,F(w_{t-s})\,ds.$$

The functions $t \to w_t$ and F are continuously differentiable. It follows that the function

$$t \to \int_0^t S(t-s)\,F(w_s)\,ds$$

is continuously differentiable and

$$\frac{d}{dt}\int_0^t S(t-s)\,F(w_s)\,ds = S(t)\,F(\varphi) + \int_0^t S(t-s)\,F'(w_s)\,v_s ds.$$

We deduce that

$$S(t)\,F(\varphi) = \frac{d}{dt}\int_0^t S(t-s)\,F(w_s)\,ds - \int_0^t S(t-s)\,F'(w_s)\,v_s ds.$$

Consequently w satisfies, for $t \in [0, T_1]$

$$w(t) = S'(t)\varphi(0) + S(t)F(\varphi) + \int_0^t S(t-s)F'(u_s)v_s ds.$$

This implies that

$$\begin{aligned} w(t) = S'(t)\,\varphi(0) &+ \frac{d}{dt}\textstyle\int_0^t S(t-s)\,F(w_s)\,ds \\ &- \textstyle\int_0^t S(t-s)\,F'(w_s)\,v_s ds + \int_0^t S(t-s)\,F'(u_s)\,v_s ds. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} u(t) - w(t) &= \frac{d}{dt}\textstyle\int_0^t S(t-s)\left(F(u_s) - F(w_s)\right)ds \\ &\quad - \textstyle\int_0^t S(t-s)\left(F'(u_s) - F'(w_s)\right)v_s ds. \end{aligned}$$

Then we deduce, for $t \in [0, T_1]$, that

$$|u_t - w_t| \le M e^{\omega T_1}\left(\int_0^t |F(u_s) - F(w_s)|\,ds + \int_0^t |F'(u_s) - F'(w_s)|\,|v_s|\,ds\right).$$

Let

$$\rho = \max\left(\sup_{-r\le s\le T_1}|u(s)|\,,\ \sup_{-r\le s\le T_1}|v(s)|\,,\ \sup_{-r\le s\le T_1}|w(s)|\right).$$

There exist $C_0(\rho), C_1(\rho) \ge 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{C}_E$ and $|\varphi_1|, |\varphi_2| \le \rho$, then

$$\begin{cases} |F(\varphi_1) - F(\varphi_2)| \le C_0(\rho)\,|\varphi_1 - \varphi_2|\,, \\ |F'(\varphi_1) - F'(\varphi_2)| \le C_1(\rho)\,|\varphi_1 - \varphi_2|\,. \end{cases}$$

This implies

$$|u_t - w_t| \le M e^{\omega T_1}\left(C_0(\rho) + \rho C_1(\rho)\right)\int_0^t |u_s - w_s|\,ds.$$

By the Gronwall's Lemma, we deduce that $u = w$ in $[0, T_1]$.

4. **Stability and symptotic behavior**

In this section, we give a result of linearized stability near an equilibrium point. By an equilibrium, we mean a constant solution. Without loss of generality, we assume that 0 is an equilibrium. We keep the assumption $(\mathbf{H}_1)$ as in Section 3 and instead of $(\mathbf{H}_2)$, we make the following hypothesis

$(\mathbf{H}_2')$: F is continuously differentiable, $F(0) = 0$ and F is globally Lipschitz continuous on $\mathcal{C}_E$, i.e. for some positive constant α, one has

$$|F(\varphi_1) - F(\varphi_2)| \leq \alpha\, |\varphi_1 - \varphi_2|\,, \text{ for all } \varphi_1, \varphi_2 \in \mathcal{C}_E.$$

From Theorem 5.2 and the Gronwall's Lemma, Condition $(\mathbf{H}_2')$ implies that, for all $\varphi \in \mathcal{C}_E$ such that $\varphi(0) \in \overline{D(A)}$, Eq.(1.1) has a unique integral solution which is defined on $[0, +\infty[$ by

$$u(t,\varphi) = S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)F(u_s(.,\varphi))ds, \quad \text{for } t \geq 0. \tag{4.1}$$

Let X denote the phase space of Eq.(1.1) defined by

$$X = \left\{\varphi \in \mathcal{C}_E : \ \varphi(0) \in \overline{D(A)}\right\}.$$

Define the following operator $\widetilde{U}(t)$ on X for $t \geq 0$ by

$$\widetilde{U}(t)(\varphi) = u_t(.,\varphi),$$

where $u(.,\varphi)$ is the unique integral solution of Eq.(1.1). Using the integral equation (6.2), it's easy to prove the following result.

Proposition 4.1. *The family* $\left(\widetilde{U}(t)\right)_{t\geq 0}$ *is a strongly continuous semigroup on* X, *that is*

(**i**) $\widetilde{U}(0) = I$,

(**ii**) $\widetilde{U}(t+s) = \widetilde{U}(t)\widetilde{U}(s)$, *for all* $t, s \geq 0$,

(**iii**) *for all* $\varphi \in X$, $\widetilde{U}(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* X,

(**iv**) *for all* $t \geq 0$, $\widetilde{U}(t)$ *is continuous from* X *into* X.

(**v**) $\left(\widetilde{U}(t)\right)_{t\geq 0}$ *satisfies, for* $t \geq 0$ *and* $\theta \in [-\tau, 0]$ *the translation property*

$$\left(\widetilde{U}(t)(\varphi)\right)(\theta) = \begin{cases} \left(\widetilde{U}(t+\theta)(\varphi)\right)(0), & \text{if } t+\theta \geq 0 \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0, \end{cases}$$

(**vi**) *there exit* $\gamma > 0$ *and* $M \geq 0$ *such that,*

$$\left|\widetilde{U}(t)(\varphi_1) - \widetilde{U}(t)(\varphi_2)\right| \leq Me^{\gamma t}\,|\varphi_1 - \varphi_2|\,, \text{ for all } \varphi_1, \varphi_2 \in X.$$

Consider the linearized equation of (1.1) corresponding to the derivative $F'(0)$ at 0

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F'(0)u_t, \ t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_E, \end{cases} \tag{4.2}$$

and let $(U(t))_{t\geq 0}$ be the corresponding solution semigroup on X.

Proposition 4.2. *The derivative at zero of the nonlinear semigroup $\widetilde{U}(t)$, $t \geq 0$, associated to Eq.(1.1), is the linear semigroup $U(t)$, $t \geq 0$, associated to Eq.(4.2).*

Proof. It enough to show that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| \leq \varepsilon \left|\varphi\right|, \text{ for } \left|\varphi\right| \leq \delta.$$

We have

$$\begin{aligned}\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| &= \sup_{\theta\in[-r,0]} \left|\left(\widetilde{U}(t)(\varphi)\right)(\theta) - \left(U(t)\varphi\right)(\theta)\right|, \\ &= \sup_{\substack{\theta\in[-r,0]\\ t+\theta\geq 0}} \left|\left(\widetilde{U}(t)(\varphi)\right)(\theta) - \left(U(t)\varphi\right)(\theta)\right|,\end{aligned}$$

and, for $t+\theta \geq 0$

$$\left(\widetilde{U}(t)(\varphi)\right)(\theta) - \left(U(t)\varphi\right)(\theta) = \frac{d}{dt}\int_0^{t+\theta} S(t+\theta-s)\left(F(\widetilde{U}(s)(\varphi)) - F'(0)(U(s)\varphi)\right) ds,$$

It follows that

$$\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| \leq Me^{\omega t}\int_0^t e^{-\omega s}\left|F(\widetilde{U}(s)(\varphi)) - F'(0)(U(s)\varphi)\right| ds$$

and

$$\begin{aligned}\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| \leq\ & Me^{\omega t}\left(\int_0^t e^{-\omega s}\left|F(\widetilde{U}(s)(\varphi)) - F(U(s)\varphi))\right| ds\right. \\ &\left. + \int_0^t e^{-\omega s}\left|F(U(s)(\varphi)) - F'(0)(U(s)\varphi)\right| ds\right).\end{aligned}$$

By virtue of the continuous differentiability of F at 0, we deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^t e^{-\omega s}\left|F(U(s)\varphi) - F'(0)(U(s)\varphi)\right| ds \leq \varepsilon\left|\varphi\right|, \quad \text{for } \left|\varphi\right| \leq \delta.$$

On the other hand, we obtain

$$\int_0^t e^{-\omega s}\left|F(\widetilde{U}(s)(\varphi)) - F(U(s)\varphi)\right| ds \leq \alpha\int_0^t e^{-\omega s}\left|\widetilde{U}(s)(\varphi) - U(s)\varphi\right| ds.$$

Consequently,

$$\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| \leq Me^{\omega t}\left(\varepsilon\left|\varphi\right| + \alpha\int_0^t e^{-\omega s}\left|\widetilde{U}(s)(\varphi) - U(s)\varphi\right| ds\right).$$

By Gronwall's Lemma, we obtain

$$\left|\widetilde{U}(t)(\varphi) - U(t)\varphi\right| \leq M\varepsilon\left|\varphi\right| e^{(\alpha M+\omega)t}.$$

We conclude that $\widetilde{U}(t)$ is differentiable at 0 and $D_\varphi\widetilde{U}(t)(0) = U(t)$, for each $t \geq 0$.

Definition 4.3. Let Y be a Banach space, $(V(t))_{t\geq 0}$ a strongly continuous semigroup of operators $V(t) : W \subseteq Y \to W$, $t \geq 0$, and $x_0 \in W$ an equilibrium of $(V(t))_{t\geq 0}$ (i.e., $V(t)x_0 = x_0$, for all $t \geq 0$). The equilibrium x_0 is called exponentially asymptotically stable if there exist $\delta > 0, \mu > 0, k \geq 1$ such that

$$\left|V(t)x - x_0\right| \leq ke^{-\mu t}\left|x - x_0\right| \text{ for all } x \in W \text{ with } \left|x - x_0\right| \leq \delta \text{ and all } t \geq 0.$$

We have the following result.

Theorem 4.4. *Suppose that the zero equilibrium of* $(U(t))_{t\geq 0}$ *is exponentially asymptotically stable, then zero is exponentially asymptotically stable equilibrium of* $\left(\widetilde{U}(t)\right)_{t\geq 0}$.

The proof of this theorem is based on the following result.

Theorem 4.5. *(Desh and Schappacher* [20]*) Let* $(V(t))_{t\geq 0}$ *be a nonlinear strongly continuous semigroup of type* γ *on a subset* W *of a Banach space* Y*, i.e.*

$$|V(t)x_1 - V(t)x_2| \leq M'e^{\gamma t}\,|x_1 - x_2|\,,\ \textit{for all } x_1, x_2 \in W$$

and assume that $x_0 \in W$ *is an equilibrium of* $(V(t))_{t\geq 0}$ *such that* $V(t)$ *is Fréchet-differentiable at* x_0 *for each* $t \geq 0$*, with* $T(t)$ *the Fréchet-derivative at* x_0 *of* $V(t)$*,* $t \geq 0$*. Then,* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup of bounded linear operators on* Y*. If the zero equilibrium of* $(T(t))_{t\geq 0}$ *is exponentially asymptotically stable, then* x_0 *is an exponentially asymptotically stable equilibrium of* $(V(t))_{t\geq 0}$.

5. Spectral analysis and characteristic equation

In this section, we consider the following linear partial functional differential equation

$$\left\{\begin{array}{l} \dfrac{du}{dt}(t) = Au(t) + L(u_t),\ \ t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_E, \end{array}\right. \tag{5.1}$$

where L is a bounded linear operator from $\mathcal{C}_E$ to E. Let us introduce the part A_0 of the operator A in $\overline{D(A)}$, which is defined by

$$\left\{\begin{array}{l} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}, \\ A_0 = A \quad \text{on } \ D(A_0). \end{array}\right.$$

For the sequel, we introduce the operators $T_0(t) = S'(t)$, for $t \geq 0$.

Lemma 5.1. [13] *The operator* A_0 *is the infinitesimal generator of* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

Moreover, formula (3.1) is equivalent to the following formula

$$u(t) = \left\{\begin{array}{l} T_0(t)\varphi(0) + \lim\limits_{\lambda\to+\infty}\displaystyle\int_0^t T_0(t-s)B_\lambda L(u_s)ds,\ t \geq 0, \\ \varphi(t),\ t \in [-r, 0]\,, \end{array}\right. \tag{5.2}$$

where $B_\lambda = \lambda(\lambda I - A)^{-1}$. Let $(U(t))_{t\geq 0}$ be the solution semigroup associated to Eq. (5.1). In order to study the asymptotic behavior of solutions, we have first to compute the infinitesimal generator A_U of $(U(t))_{t\geq 0}$.

Theorem 5.2. *The infinitesimal generator* A_U *of the semigroup* $(U(t))_{t\geq 0}$ *on* X *is given by*

$$\left\{\begin{array}{ll} D(A_U) = & \left\{\varphi \in C^1([-r,0]\,; E) : \varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \right. \\ & \left. \textit{and } \varphi'(0) = A\varphi(0) + L(\varphi)\right\}, \\ & \\ A_U\varphi = \varphi' & \end{array}\right.$$

Proof. Let A_U be the infinitesimal generator of the semigroup $(U(t))_{t\geq 0}$ on X and let $\varphi \in D(A_U)$. Then

$$\begin{cases} \lim\limits_{t\to 0^+} \dfrac{U(t)\varphi - \varphi}{t} = \psi \ \text{ exists in } X, \\ A_U\varphi = \psi. \end{cases}$$

The first expression implies that

$$\lim_{t\to 0^+} \frac{\varphi(t+\theta) - \varphi(\theta)}{t} = \psi(\theta), \quad \text{for } \theta \in [-r, 0)\,.$$

On the other hand, we have

$$\lim_{t\to 0^+} \frac{\varphi(t+\theta) - \varphi(\theta)}{t} = D^+\varphi(\theta), \quad \text{for } \theta \in [-r, 0)\,,$$

where $D^+\varphi$ is the right derivative of the function φ. Then, $D^+\varphi = \psi$ exists and is continuous on $[-r, 0)$. For the next, we need to use following lemma.

Lemma 5.3. *(*[41]*, Corollary 1.2, p. 43) Let φ be a continuous and right differentiable function on $[a, b)$. If the function $D^+\varphi$ is continuous on $[a, b)$, then φ is continuously differentiable on $[a, b)$.*

We deduce from this lemma that the function φ is continuously differentiable on $[-r, 0)$ and $\varphi' = \psi$ on $[-r, 0)$. Note that $\psi \in X$. Then, $\lim\limits_{\theta\to 0} \varphi'(\theta) = \psi(0)$ exists. This proves that the function φ is continuously differentiable on $[-r, 0]$ and $\varphi' = \psi$. On the other hand, as $\varphi \in D(A_U)$, the semigroup $t \to U(t)\varphi$ is differentiable. This implies that the integral solution $u : t \mapsto (U(t)\varphi)(0)$ of Eq.(1.1) is continuously differentiable on $[0, +\infty)$. By Proposition 3.3, we deduce that u is a strict solution of Eq.(1.1). Then, we obtain

$$\lim_{t\to 0^+} \frac{u(t) - \varphi(0)}{t} = u'(0) = \varphi'(0) \quad \text{and} \quad u'(0) = Au(0) + L(u_0).$$

Consequently,

$$\varphi'(0) = A\varphi(0) + L(\varphi).$$

We have proved that

$$\begin{cases} D(A_U) \subseteq \begin{array}{l} \{\varphi \in C^1([-r, 0]\,; E) : \varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \\ \text{and } \varphi'(0) = A\varphi(0) + L(\varphi)\}\,, \end{array} \\ \\ A_U\varphi = \varphi' \end{cases}$$

Consider now $\varphi \in C^1([-r, 0]\,; E)$ such that

$$\varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \quad \text{and} \quad \varphi'(0) = A\varphi(0) + L(\varphi).$$

Let $u : [-r, +\infty) \to E$ be the unique integral solution of Eq.(1.1). We have

$$u(t) = \begin{cases} (U(t)\varphi)(0),\ t > 0, \\ \varphi(t),\ t \in [-r, 0]\,. \end{cases}$$

From Theorem 3.5, we deduce that u is a strict solution. This implies also that $t \mapsto u_t$ is continuously differentiable on $[0, +\infty)$. Hence $\varphi \in D(A_U)$.

In the sequel of this work, we suppose that

(**H3**) The semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$ is compact, this means that for each $t>0$, the operator $T_0(t)$ is compact on $\overline{D(A)}$.

Theorem 5.4. *Assume that* (**H3**) *holds. Then, the semigroup* $(U(t))_{t\geq 0}$ *is compact on* X*, for every* $t>r$*.*

Proof. Let B be a bounded subset of X and let $t>r$. We will use the Ascoli-Arzela's theorem to show that the set $\{U(t)\varphi : \varphi\in B\}$is relatively compact in X. It is enough to show that, for every $\theta\in[-r,0]$, $\{(U(t)\varphi)(\theta):\varphi\in B\}$ is relatively compact in E. Let $\varepsilon>0$ such that $t+\theta-\varepsilon>0$. Then, using formula (5.2), we obtain

$$(U(t)\varphi)(\theta)=T_0(t+\theta)\varphi(0)+\lim_{\lambda\to+\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi))ds.$$

Note that

$$\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds=\int_0^{t+\theta-\varepsilon}T_0(t+\theta-s)B_\lambda L(U(s)\varphi))ds+ \int_{t+\theta-\varepsilon}^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi))ds$$

and

$$\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon}T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds= T_0(\varepsilon)\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon}T_0(t+\theta-\varepsilon-s)B_\lambda L(U(s)\varphi))ds.$$

Assumption (**H3**) implies that the set

$$T_0(\varepsilon)\left\{\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon}T_0(t+\theta-\varepsilon-s)B_\lambda L(U(s)\varphi))ds:\varphi\in B\right\}$$

is relatively compact in E. On other hand, the semigroup $(U(t))_{t\geq 0}$ is bounded. Then, for some positive constant b_1, we have

$$\left|\lim_{\lambda\to+\infty}\int_{t+\theta-\varepsilon}^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi))ds\right|\leq b_1\varepsilon.$$

Consequently, the set

$$\left\{\lim_{\lambda\to+\infty}\int_{t+\theta-\varepsilon}^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi))ds:\varphi\in B\right\}$$

is totally bounded. Therefore, the set $\{(U(t)\varphi)(\theta):\varphi\in B\}$ is relatively compact. We show now the equicontinuity. Let $\theta>\theta_0$, we have

$$(U(t)\varphi)(\theta)-(U(t)\varphi)(\theta_0)=(T_0(t+\theta)-T_0(t+\theta_0))\varphi(0)+ \lim_{\lambda\to+\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds- \lim_{\lambda\to+\infty}\int_0^{t+\theta_0}T_0(t+\theta_0-s)B_\lambda L(U(s)\varphi)ds.$$

On the other hand, we have

$$\begin{aligned}\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds &= \int_0^{t+\theta_0} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds \\ &\quad + \int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds.\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}\left|\left(U(t)\varphi\right)(\theta)-\left(U(t)\varphi\right)(\theta_0)\right| &\le \left|T_0(t+\theta)-T_0(t+\theta_0)\right|\left|\varphi(0)\right| + \\ &\left|\lim_{\lambda\to+\infty}\int_0^{t+\theta_0}\left(T_0(t+\theta-s)-T_0(t+\theta_0-s)\right)B_\lambda L(U(s)\varphi)ds\right| + \\ &\left|\lim_{\lambda\to+\infty}\int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds\right|.\end{aligned}$$

From Assumption (**H3**), we deduce that the semigroup $(T_0(t))_{t\ge0}$ is uniformly continuous for $t>0$. Then,

$$\lim_{\theta\to\theta_0}\left|T_0(t+\theta)-T_0(t+\theta_0)\right| = 0.$$

Furthermore, the semigroup $(U(t))_{t\ge0}$ is bounded in B. Consequently, there exists a positive constant b_2 such that

$$\left|\lim_{\lambda\to+\infty}\int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds\right| \le b_2\left(\theta-\theta_0\right)$$

and

$$\begin{aligned}\lim_{\lambda\to+\infty}\int_0^{t+\theta_0}\left(T_0(t+\theta-s)-T_0(t+\theta_0-s)\right)B_\lambda L(U(s)\varphi))ds &= \\ \left(T_0(\theta-\theta_0)-I\right)\lim_{\lambda\to+\infty}\int_0^{t+\theta_0} T_0(t+\theta_0-s)B_\lambda L(U(s)\varphi)ds.\end{aligned}$$

Then, there exists a compact set K in E such that

$$\lim_{\lambda\to+\infty}\int_0^{t+\theta_0} T_0(t+\theta_0-s)B_\lambda L(U(s)\varphi)ds \in K, \text{ uniformly in } \varphi\in B.$$

By the Banach-Steinhaus's theorem, we obtain

$$\lim_{\theta\to\theta_0}\sup_{x\in K}\left|(T_0(\theta-\theta_0)-I)x\right| = 0.$$

This implies that

$$\lim_{\theta\to\theta_0^+}\left(U(t)\varphi\right)(\theta)-\left(U(t)\varphi\right)(\theta_0)=0, \quad \text{uniformly in } \varphi\in B.$$

Using a similar argument as above, we prove that

$$\lim_{\theta\to\theta_0^-}\left(U(t)\varphi\right)(\theta)-\left(U(t)\varphi\right)(\theta_0)=0, \ \text{uniformly in } \varphi\in B.$$

The Ascoli-Arzela's theorem completes the proof of Theorem 5.4.

We now consider the spectral properties of the generator A_U. For each complex number λ, we define the linear operator $\Delta(\lambda): D(A)\to E$ by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda\cdot}I), \tag{5.3}$$

where $e^{\lambda\cdot}I : E \to \mathcal{C}_E$, is defined by (note that we consider here the complexification of $\mathcal{C}_E$)

$$\left(e^{\lambda\cdot}x\right)(\theta) = e^{\lambda\theta}x,\ x \in E \quad \text{and} \quad \theta \in [-r, 0]\,.$$

We say that λ is a characteristic value of Eq.(1.1) if there exists $x \in D(A)\backslash\{0\}$ solving the characteristic equation $\Delta(\lambda)x = 0$. By [21], and from the compactness property of the semigroup $(U(t))_{t\geq 0}$, we get the following result.

Corollary 5.5. *Assume that* (**H3**) *holds. Then, for each* $t > r$, *the spectrum* $\sigma(U(t))$ *is a countable set and is compact with the only possible accumulation point* 0 *and if* $\mu \neq 0 \in \sigma(U(t))$ *then* $\mu \in P\sigma(U(t))$, *where* $P\sigma(U(t)$ *denotes the point spectrum.*

Corollary 5.6. *Assume that* (**H3**) *holds. Then, there exists a real number* δ *such that* $\operatorname{Re}\lambda \leq \delta$ *for all* $\lambda \in \sigma(A_U)$. *Moreover, if* β *is a given real number then there exists only a finite number of* $\lambda \in P\sigma(A_U)$ *such that* $\operatorname{Re}\lambda > \beta$.

We can give now an exponential estimate for the semigroup solution.

Proposition 5.7. *Assume that* (**H3**) *holds. Let* δ *be a real number such that* $\operatorname{Re}\lambda \leq \delta$ *for all characteristic values* λ *of Eq.(1.1). Then, for* $\gamma > 0$ *there exists a constant* $k(\gamma) \geq 1$ *such that*

$$|U(t)\varphi| \leq k(\gamma)e^{(\delta+\gamma)t}\,|\varphi|\,,\ \textit{for all}\ t \geq 0\ \textit{and}\ \varphi \in X.$$

Proof. Let ω_0 be defined by

$$\omega_0 := \inf\left\{\omega \in \mathbb{R} :\ \sup_{t\geq 0} e^{-\omega t}\,|U(t)| < +\infty\right\}.$$

The compactness property of the semigroup (see [21]), implies that

$$\omega_0 = s_1(A_U) := \sup\left\{\operatorname{Re}\lambda :\ \lambda \in P\sigma(A_U)\right\}.$$

On the other hand, if $\lambda \in P\sigma(A_U)$ then there exists $\varphi \neq 0 \in D(A_U)$ such that $A_U\varphi = \lambda\varphi$. This implies that

$$\varphi(\theta) = e^{\lambda t}\varphi(0) \ \text{ and } \ \varphi'(0) = A\varphi(0) - L(\varphi) \ \text{ with } \ \varphi(0) \neq 0.$$

Then, $\Delta(\lambda)\varphi(0) = 0$. We deduce that λ is a characteristic value of Eq.(1.1).
We will prove now the existence of $\lambda \in P\sigma(A_U)$ such that $\operatorname{Re}\lambda = s_1(A_U)$. Let $(\lambda_n)_n$ be a sequence in $P\sigma(A_U)$ such that $\operatorname{Re}\lambda_n \to s_1(A_U)$ as $n \to +\infty$. Then, there exists β such that $\operatorname{Re}\lambda_n > \beta$ for $n \geq n_0$ with n_0 large enough. From Corollary 5.6, we deduce that $\{\lambda_n : \operatorname{Re}\lambda_n > \beta\}$ is finite. The sequence $(\operatorname{Re}\lambda_n)_n$ is stationary. Consequently, there exists n such that $\operatorname{Re}\lambda_n = s_1(A_U)$.

The asymptotic behavior of the solutions is now completely obtained by the characteristic equation.

Theorem 5.8. *Assume that* (**H3**) *holds. Let* δ *be the smallest real number such that if* λ *is any characteristic value of Eq.(1.1), then* $\operatorname{Re}\lambda \leq \delta$. *If* $\delta < 0$, *then for all* $\varphi \in X$, $|U(t)\varphi| \to 0$ *as* $t \to +\infty$. *If* $\delta = 0$ *then there exists* $\varphi \in X\backslash\{0\}$ *such that* $|U(t)\varphi| = |\varphi|$ *for all* $t \geq 0$. *If* $\delta > 0$, *then there exists* $\varphi \in X$ *such that* $|U(t)\varphi| \to +\infty$ *as* $t \to +\infty$.

Proof. Assume that $\delta < 0$, then we have $\omega_0 = s_1(A_U) < 0$ and the stability holds. If $\delta = 0$, then there exists $x \neq 0$ and a complex λ such that $\operatorname{Re}\lambda = 0$ and $\Delta(\lambda)x = 0$. Then, $\lambda \in P\sigma(A_U)$ and $e^{\lambda t} \in P\sigma(U(t))$. Consequently, there exists $\varphi \neq 0$ such that

$$U(t)\varphi = e^{\lambda t}\varphi.$$

This implies that $|U(t)\varphi| = \left|e^{\lambda t}\varphi\right| = |\varphi|$. Assume now that $\delta > 0$. Then, there exists $x \neq 0$ and a complex λ such that $\operatorname{Re}\lambda = \delta$ and $\Delta(\lambda)x = 0$. Then, there exists $\varphi \neq 0$ such that $|U(t)\varphi| = e^{\delta t}|\varphi| \to +\infty$, as $t \to +\infty$.

6. A Variation of constants formula

In this section, we consider the following nonhemegeneous linear partial functional differential equation

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + L(u_t) + f(t), \text{ for } t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_E, \end{cases} \tag{6.1}$$

where f is a continuous function form $\mathbb{R}$ to E. To construct a variation of constants formula associated to Eq.(6.1), we define the space $X \oplus \langle X_0\rangle$, where $\langle X_0\rangle$ is the space given by

$$\langle X_0\rangle = \{X_0c : c \in E\}$$

and the function X_0c is defined by

$$(X_0c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ c & \text{if } \theta = 0. \end{cases}$$

$X \oplus \langle X_0\rangle$ is endowed with the following norm

$$|\varphi + X_0c| = |\varphi| + |c|.$$

Theorem 6.1. *The continuous extension $\widetilde{A_U}$ of the operator A_U defined on $X \oplus \langle X_0\rangle$ by*

$$\begin{cases} D(\widetilde{A_U}) = \left\{\varphi \in C^1([-r, 0]; E) : \varphi(0) \in D(A) \text{ and } \varphi'(0) \in \overline{D(A)}\right\}, \\ \widetilde{A_U}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi'(0)), \end{cases}$$

is a Hille-Yosida operator.

The proof of this theorem is based on the following lemma.

Lemma 6.2. *There exists ω_1 such that for $\lambda > \omega_1$,*
(i) *$\Delta(\lambda)$ is invertible for $\lambda > \omega_1$ and $|\Delta^{-1}(\lambda)| \leq \dfrac{M}{\lambda - \omega_1}$, for $\lambda > \omega_1$,*
(ii) *$D(\widetilde{A_U}) = D(A_U) \oplus \langle e^{\lambda \cdot}\rangle$, where $\langle e^{\lambda \cdot}\rangle = \{e^{\lambda \cdot}c : c \in D(A) \text{ and } (e^{\lambda \cdot}c)(\theta) = e^{\lambda\theta}c\}$,*
(iii) *$(\omega_1, +\infty) \subset \rho(\widetilde{A_U})$ and for $n \geq 1$*

$$R(\lambda, \widetilde{A_U})^n(\varphi + X_0c) = R(\lambda, A_U)^n\varphi + R(\lambda, A_U)^{n-1}\left(e^{\lambda \cdot}\Delta^{-1}(\lambda)c\right),$$

for every $(\varphi, c) \in X \times E$, where $R(\lambda, A_U) = (\lambda I - A_U)^{-1}$ and $R(\lambda, \widetilde{A_U}) = \left(\lambda I - \widetilde{A_U}\right)^{-1}$.

Proof of the lemma. **(i)** We have

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I) = (\lambda I - A)\left(I - R(\lambda, A)L(e^{\lambda \cdot} I)\right).$$

Let $\omega_0 > \max(0, \omega)$. Then,

$$\left|R(\lambda, A)L(e^{\lambda \cdot} I)\right| \leq \frac{M\,|L|}{\lambda - \omega_0} < 1, \text{ for } \lambda > \omega_0 + M\,|L|.$$

Hence, the operator $I - R(\lambda, A)L(e^{\lambda \cdot} I)$ is invertible and satisfies

$$\left(I - R(\lambda, A)L(e^{\lambda \cdot} I)\right)^{-1} = \sum_{n \geq 0} \left[R(\lambda, A)L(e^{\lambda \cdot} I)\right]^n.$$

Then,

$$\left|\left(I - R(\lambda, A)L(e^{\lambda \cdot} I)\right)^{-1}\right| \leq \frac{1}{1 - |R(\lambda, A)L(e^{\lambda \cdot} I)|}$$

and

$$\left|\left(I - R(\lambda, A)L(e^{\lambda \cdot} I)\right)^{-1}\right| \leq \frac{1}{1 - \frac{M|L|}{\lambda - \omega_0}} = \frac{\lambda - \omega_0}{\lambda - \omega_0 - M\,|L|}.$$

This implies that

$$\left|\Delta^{-1}(\lambda)\right| \leq \frac{M}{\lambda - \omega_0 - M\,|L|}, \qquad \text{for } \lambda > \omega_0 + M\,|L|.$$

(ii) For $\lambda > \omega_1 := \omega_0 + M\,|L|$, it is clear that $D(A_U) \cap \left\langle e^{\lambda \cdot} \right\rangle = \{0\}$. Let $\widetilde{\psi} \in D(\widetilde{A_U})$. If we put $\psi = \widetilde{\psi} - e^{\lambda \cdot} \Delta(\lambda)^{-1}\left(A\widetilde{\psi}(0) + L(\widetilde{\psi}) - \widetilde{\psi}'(0)\right)$, then we deduce that $\psi \in D(A)$. This implies that $D(\widetilde{A_U}) = D(A_U) + \left\langle e^{\lambda \cdot} \right\rangle$ and the decomposition holds.

(iii) Let $\widetilde{\varphi} \in X \oplus \langle X_0 \rangle$, $\widetilde{\varphi} = \varphi + X_0 c$. We are looking for $\widetilde{\psi} \in D(\widetilde{A_U})$, such that $(\lambda I - \widetilde{A_U})\widetilde{\psi} = \widetilde{\varphi}$. The function $\widetilde{\psi}$ can be written as $\widetilde{\psi} = \psi + e^{\lambda \cdot} a$, where $\psi \in D(A_U)$ and $a \in D(A)$. Then we have

$$(\lambda I - \widetilde{A_U})(\psi + e^{\lambda \cdot} a) = \varphi + X_0 c.$$

This equation slips into two equations

$$\begin{cases} (\lambda I - A_U)\psi = \varphi, \\ \Delta(\lambda) a = c. \end{cases}$$

It follows that for $\lambda > \omega_1$, $(\lambda I - \widetilde{A_U})^{-1}$ exists and

$$(\lambda I - \widetilde{A_U})^{-1}(\varphi + X_0 c) = (\lambda I - A_U)^{-1}\varphi + e^{\lambda \cdot}\Delta(\lambda)^{-1} c.$$

Repeating this procedure, we obtain for every $n \geq 1$

$$R(\lambda, \widetilde{A_U})^n(\varphi + X_0 c) = R(\lambda, A_U)^n \varphi + R(\lambda, A_U)^{n-1}\left(e^{\lambda \cdot}\Delta^{-1}(\lambda) c\right). \tag{6.2}$$

From **(i)**, we deduce that

$$\sup_{n \in \mathbb{N},\ \lambda > \omega_1} \left|(\lambda - \omega_1)^n R(\lambda, \widetilde{A}_U)^n\right| < +\infty.$$

This completes the proof of the lemma and the theorem.

Lemma 6.3. *The part of* $\widetilde{A_U}$ *in* $\overline{D(\widetilde{A_U})}$ *is the operator* A_U*.*

Proof. The operator A_U generates a strongly continuous semigroup in X. Then, by the Hille-Yosida theorem we have $\overline{D(A_U)} = X$. On the other hand, $D(A_U) \subset D(\widetilde{A_U}) \subset X$. This implies that

$$\overline{D(\widetilde{A_U})} = \left\{\varphi \in \mathcal{C}_E : \varphi(0) \in \overline{D(A)}\right\} = X.$$

Let $\widetilde{B}$ be the part of $\widetilde{A_U}$ in $\overline{D(\widetilde{A_U})}$. Then

$$\begin{cases} D(\widetilde{B}) = \left\{\varphi \in D(\widetilde{A_U}) : \widetilde{A_U}\varphi \in X\right\}, \\ \widetilde{B}\varphi = \widetilde{A_U}\varphi. \end{cases}$$

$\varphi \in D(\widetilde{B})$ if and only if $\varphi \in C^1([-r,0];E)$, $\varphi(0) \in D(A)$, $\varphi'(0) \in \overline{D(A)}$ and $\varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi'(0)) \in X$, which is equivalent to say that $\varphi \in D(A_U)$. Consequently, we have $\widetilde{B} = A_U$.

Consider now the following nonhomogeneous Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) = \widetilde{A_U}u(t) + X_0 f(t), \text{ for } t \geq 0 \\ u(0) = \varphi \in \mathcal{C}_E. \end{cases} \tag{6.3}$$

Definition 6.4. A continuous function $u : [0,+\infty) \to \mathcal{C}_E$ is called an integral solution of Eq.(6.3) if

(i) $\int_0^t u(s)ds \in D(\widetilde{A_U})$, for $t \geq 0$,

(ii) $u(t) = \varphi + \widetilde{A_U}\int_0^t u(s)ds + X_0\int_0^t f(s)ds$, for $t \geq 0$.

Applying Theorem 2.10, we conclude that for all $\varphi \in X$ Eq.(6.3) has a unique integral solution which is given by the following formula

$$u(t) = U(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t U(t-s)\widetilde{B_\lambda}X_0 f(s)ds, \text{ for } t \geq 0, \tag{6.4}$$

where $\widetilde{B_\lambda} = \lambda(\lambda I - \widetilde{A_U})^{-1}$.

Theorem 6.5. *Let x be an integral solution of Eq.(6.1). Then, the function u given by*

$$u(t) = x_t, \quad \textit{for } t \geq 0,$$

is the unique integral solution of (6.3). Conversely, if u is an integral solution of Eq.(6.3), then the function x defined by

$$x(t) = \begin{cases} u(t)(0), & \textit{if } t \geq 0, \\ \varphi(t), & \textit{if } t \leq 0, \end{cases}$$

is an integral solution of Eq.(6.1).

Proof. Let $\varphi \in X$ and x be the integral solution of Eq.(6.1). It suffices to show that the function u defined by

$$u(t) = x_t, \text{ for } t \geq 0, \tag{6.5}$$

is an integral solution of Eq.(6.3). Let $t \geq 0$. Then,

$$\frac{d}{d\theta}\left(\int_0^t x_s\,ds\right) = x_t - \varphi.$$

We deduce that

$$\widetilde{A_U}\left(\int_0^t x_s ds\right) = x_t - \varphi + X_0\left(A\int_0^t x(s)ds + L\left(\int_0^t x_s ds\right) - x(t) + \varphi(0)\right).$$

On the other hand, x is given by

$$x(t) = \varphi(0) + A\int_0^t x(s)ds + L\left(\int_0^t x_s ds\right) + \int_0^t f(s)ds.$$

It follows that

$$u(t) = \varphi + \widetilde{A_U}\int_0^t u(s)ds + X_0\int_0^t f(s)ds.$$

Then, u is an integral solution of Eq.(6.3) and we have the following formula

$$u(t) = U(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t U(t-s)\widetilde{B_\lambda}X_0 f(s)ds, \quad \text{for } t \geq 0.$$

Let now u be an integral solution of Eq.(6.3). Then, u satisfies the following translation property

$$u(t)(\theta) = \begin{cases} u(t+\theta)(0)\,, \text{ if } t+\theta \geq 0, \\ \varphi(t+\theta), \text{ if } t+\theta \leq 0. \end{cases}$$

In fact, if $t+\theta \geq 0$, we have

$$u(t)(\theta) = \left(U(t)\varphi\right)(\theta) + \lim_{\lambda\to+\infty}\int_0^t \left(U(t-s)\widetilde{B_\lambda}X_0 f(s)\right)(\theta)ds.$$

Then

$$u(t)(\theta) = \left(U(t+\theta)\varphi\right)(0) + \lim_{\lambda\to+\infty}\int_0^{t+\theta}\left(U(t+\theta-s)\widetilde{B_\lambda}X_0 f(s)\right)(0)ds +$$
$$\lim_{\lambda\to+\infty}\int_{t+\theta}^t \left(U(t-s)\widetilde{B_\lambda}X_0 f(s)\right)(\theta)ds.$$

This implies that

$$u(t)(\theta) = u(t+\theta)(0) + \lim_{\lambda\to+\infty}\int_{t+\theta}^t\left(U(t-s)\widetilde{B_\lambda}X_0 f(s)\right)(\theta)ds.$$

Furthermore,

$$\lim_{\lambda\to+\infty}\int_{t+\theta}^t\left(\widetilde{B_\lambda}X_0 f(s)\right)(t-s+\theta)ds = \lim_{\lambda\to+\infty}\int_{t+\theta}^t e^{\lambda(t-s+\theta)}\lambda\Delta^{-1}(\lambda)f(s)ds.$$

Which gives that

$$\lim_{\lambda\to+\infty}\int_{t+\theta}^t e^{\lambda(t-s+\theta)}\lambda\Delta^{-1}(\lambda)f(s)ds = 0.$$

The translation property implies that $x_t = u(t)$, for $t \geq 0$. Consequently,

$$x_t = \varphi + \widetilde{A_U}\int_0^t x_s ds + X_0\int_0^t f(s)ds, \quad \text{for } t \geq 0.$$

As above, we get that x is an integral solution of Eq.(6.1).

7. Hyperbolicity and Boundedness

In this section, we are concerned with the existence of bounded solutions of Eq.(6.1) in hyperbolic case.

Definition 7.1. We say that the semigroup $(U(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(A_U)\cap i\mathbb{R}=\emptyset.$$

From the compactness of the semigroup $(U(t))_{t\geq 0}$ and from[21], we get the following result on the spectral decomposition of the phase space X.

Theorem 7.2. [21] *Assume that* (**H3**) *holds. If the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic, then the space* X *is decomposed as a direct sum* $X = S\oplus US$ *of two* $U(t)$ *invariant closed subspaces* S *and* US *such that the restricted semigroup on* US *is a group and there exist positive constants* $\overline{M}$ *and* γ *such that*

$$\begin{array}{l} |U(t)\varphi|\leq \overline{M}e^{-\gamma t}\,|\varphi|\,,\ t\geq 0,\ \varphi\in S\\ |U(t)\varphi|\leq \overline{M}e^{\gamma t}\,|\varphi|\,,\ t\leq 0,\ \varphi\in US \end{array}$$

We give now our first main result of this section.

Theorem 7.3. *Assume that* (**H3**) *holds and the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic. Let* B *represent* $B(\mathbb{R}^-)$, $B(\mathbb{R}^+)$ *or* $B(\mathbb{R})$, *the set of bounded continuous functions from* $\mathbb{R}^-$, $\mathbb{R}^+$ *or* $\mathbb{R}$ *respectively to* E. *Let* $\pi : B\to B$ *be a projection onto the integral solutions of Equation (6.1) (for any* $\varphi\in X$*) which are in* B. *Then, for any* $f\in B$, *there is a unique solution* $\mathcal{K}f\in B$ *of Eq.(6.1) (for some* $\varphi\in X$*) such that* $\pi\mathcal{K}f=0$ *and* $\mathcal{K}: B\to B$ *is a continuous linear operator. Moreover,*
(**i**) *for* $B=B(\mathbb{R}^-)$, *we have*

$$\pi(B)=\ \left\{x:\mathbb{R}^-\to E,\ \text{there exists } \varphi\in US \text{ such that } x(t)=\left(U(t)\varphi\right)(0),\quad t\leq 0\right\}$$

and

$$\begin{aligned}(\mathcal{K}f)_t = & \lim_{s\to-\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0f(\tau)\right)^S d\tau+\\ & \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0f(\tau)\right)^{US} d\tau.\end{aligned}$$

(**ii**) *For* $B=B(\mathbb{R}^+)$, *we have*

$$\pi(B)=\ \left\{x:\mathbb{R}^+\to E,\ \text{there exists } \varphi\in S \text{ such that } x(t)=\left(U(t)\varphi\right)(0),\quad t\geq 0\right\}$$

and

$$\begin{aligned}(\mathcal{K}f)_t = & \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0f(\tau)\right)^S d\tau+\\ & \lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0f(\tau)\right)^{US} d\tau.\end{aligned}$$

(**iii**) *For* $B=B(\mathbb{R})$, *we have*

$$\pi(B)=\{0\}$$

and

$$\begin{aligned}(\mathcal{K}f)_t = \ & \lim_{s\to-\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S d\tau + \\ & \lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau.\end{aligned}$$

Proof. Corollary 7.2 implies that $U(t)\,(US) \subseteq US$. Then,

$$\left\{x : \mathbb{R}^- \to E, \text{ there exists } \varphi \in US \text{ such that } x(t) = \left(U(t)\varphi\right)(0),\ t \leq 0\right\}$$

is a subspace of $\pi(B(\mathbb{R}^-))$. Conversely, let $\varphi \in S$ and $u(.,\varphi)$ be the integral solution of Eq.(1.1) in S, which is bounded on $\mathbb{R}^-$. Assume that there is a $t \in (-\infty, 0]$ such that $u_t(.,\varphi) \neq 0$. Then, for any $s \in (-\infty, t)$, we have

$$u_t(.,\varphi) = U(t-s)u_s(.,\varphi).$$

Thanks to Theorem 7.2 we have

$$|u_t(.,\varphi)| \leq \overline{M}e^{-\gamma(t-s)}\,|u_s(.,\varphi)|\,,\ \ s \leq t.$$

Since $u_s(.,\varphi)$ is bounded, we deduce that $u_s(.,\varphi) = 0$. Therefore,

$$\begin{aligned}\pi(B(\mathbb{R}^-)) \subseteq \ & \{x : \mathbb{R}^- \to E, \text{ there exists } \varphi \in US \\ & \qquad \text{such that } x(t) = \left(U(t)\varphi\right)(0), \quad t \leq 0\}.\end{aligned}$$

In the same manner, one can prove the same relations for $B(\mathbb{R}^+)$ and $B(\mathbb{R})$.
Let $f \in B(\mathbb{R}^-)$ and $u = u(.,\varphi, f)$ be a solution of Eq.(6.1) in $B(\mathbb{R}^-)$, with initial value $\varphi \in X$. Then, the function u can be decomposed as

$$u_t = u_t^{US} + u_t^S,$$

where $u_t^{US} \in US$ and $u_t^S \in S$ are given by

$$u_t^{US} = U(t-s)u_s^{US} + \lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau, \text{ for } t, s \in \mathbb{R}, \tag{7.1}$$

$$u_t^S = U(t-s)u_s^S + \lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S d\tau, \text{ for } s \leq t \leq 0, \tag{7.2}$$

since $U(t)$ is defined on US for all $t \in \mathbb{R}$. By Theorem 7.2, we deduce that

$$u_t^S = \lim_{s\to-\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S d\tau, \text{ for } t \leq 0. \tag{7.3}$$

By Lemma 6.2, we get $\widetilde{B_\lambda}X_0 = \lambda e^{\lambda\cdot}\Delta^{-1}(\lambda)$. Thus,

$$\left|U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S\right| \leq \overline{M}e^{-\gamma(t-\tau)}\frac{M\lambda}{\lambda-\omega_1}\sup_{\tau\in(-\infty,0]}|f(\tau)|\,.$$

Consequently, we get

$$\left|u_t^S\right| \leq \frac{M\overline{M}}{\gamma}\sup_{\tau\in(-\infty,0]}|f(\tau)|\,, t \leq 0. \tag{7.4}$$

We have proved that

$$
\begin{aligned}
u_t = \;& U(t)\varphi^{US} + \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau + \\
& \lim_{s\to-\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{S} d\tau, \ \text{ for } t\le 0.
\end{aligned}
\tag{7.5}
$$

We obtain also, for $t\le 0$, the following estimate

$$
\left|U(t)\varphi^{US} + \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau\right| \le \overline{M}e^{\gamma t}\left|\varphi^{US}\right| + \frac{M\overline{M}}{\gamma}\sup_{\tau\in(-\infty,0]}|f(\tau)|\,.
\tag{7.6}
$$

Conversely, we can verify that the expression (7.5) is a solution of Eq.(6.1) in $B(\mathbb{R}^-)$ satisfying the estimates (7.4) and (7.6) for every $\varphi\in X$. Let $u=u(.,\varphi^{US},f)$ be defined by (7.5) and let $\mathcal{K}: B(\mathbb{R}^-)\to B(\mathbb{R}^-)$ be defined by $\mathcal{K}f=(I-\pi)\,u(.,0,f)$. We can easily verify that

$$
\begin{cases}
u_t(.,\varphi^{US},0)=U(t)\varphi^{US},\\
(I-\pi)\,u(.,\varphi^{US},0)=0,\\
u(.,\varphi^{US},f)=u(.,\varphi^{US},0)+u(.,0,f).
\end{cases}
$$

Therefore, $\mathcal{K}$ is a continuous linear operator on $B(\mathbb{R}^-)$, $\mathcal{K}f$ satisfies Eq.(6.1) for every $f\in B(\mathbb{R}^-)$, $\pi\mathcal{K}=0$ and $\mathcal{K}$ is explicitly given by Theorem 7.3.

For $B(\mathbb{R}^+)$ and $B(\mathbb{R})$ the proofs are similar. This completes the proof of Theorem 7.3.

We consider now the following nonlinear partial functional differential equation

$$
\frac{du}{dt}(t)=Au(t)+L(u_t)+g(t,u_t),\ t\in\mathbb{R}
\tag{7.7}
$$

and its integrated form

$$
u_t=U(t)\varphi+\lim_{\lambda\to+\infty}\int_0^t U(t-s)\widetilde{B_\lambda}X_0 g(s,u_s)ds,
\tag{7.8}
$$

where g is continuous from $\mathbb{R}\times\mathcal{C}_E$ into E. We assume that

(H4) $g(t,0)=0$ for $t\in\mathbb{R}$ and there exists a nondecreasing function $\alpha:[0,+\infty)\to[0,+\infty)$ with $\lim\limits_{h\to 0}\alpha(h)=0$ and

$$
g(t,\varphi_1)-g(t,\varphi_2)\le\alpha(h)\left|\varphi_1-\varphi_2\right| \quad \text{for } \varphi_1,\varphi_2\in\mathcal{C}_E, |\varphi_1|\,,|\varphi_2|\le h,\ t\in\mathbb{R}.
$$

Proposition 7.4. *Assume that* (**H3**), (**H4**) *hold and the semigroup* $(U(t))_{t\ge 0}$ *is hyperbolic. Then, there exists* $h>0$ *and* $\varepsilon\in\,]0,h[$ *such that for any* $\varphi\in S$ *with* $|\varphi|\le\varepsilon$, *Eq.(7.8) has a unique bounded solution* $u:[-r,+\infty)\to E$ *with* $|u_t|\le h$ *for* $t\ge 0$ *and* $u_0^S=\varphi$.

Proof. Let $\varphi\in S$. By Theorem 7.7, it suffices to establish the existence of a bounded solution $u:[-r,+\infty)\to E$ of the following equation

$$
\begin{aligned}
u_t = \;& U(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^{S} d\tau + \\
& \lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^{US} d\tau.
\end{aligned}
$$

Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence of continuous functions from $[-r,+\infty)$ to E, defined by

$$\begin{aligned} u_t^{(0)} = &\quad U(t)\varphi \\ u_t^{(n+1)} = &\quad U(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau^{(n)})\right)^S d\tau + \\ &\quad \lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau^{(n)})\right)^{US} d\tau. \end{aligned}$$

It is clear that $\left(u_0^{(n)}\right)^S = \varphi$. Moreover, we can choose $h>0$ and $\varepsilon\in\,]0,h[$ small enough such that, if $|\varphi|\le\varepsilon$ then $\left|u_t^{(n)}\right| < h$ for $t\ge 0$.
On the other hand, we have

$$\begin{aligned} \left|u_t^{(n+1)} - u_t^{(n)}\right| \ &\le \int_0^t \overline{M}Me^{-\gamma(t-\tau)}\alpha(h)\left|u_\tau^{(n)} - u_\tau^{(n-1)}\right| d\tau + \\ &\int_t^{+\infty} \overline{M}Me^{\gamma(t-\tau)}\alpha(h)\left|u_\tau^{(n)} - u_\tau^{(n-1)}\right| d\tau. \end{aligned}$$

By induction we get

$$\left|u_t^{(n+1)} - u_t^{(n)}\right| \le 2h\left(\frac{2\overline{M}M\alpha(h)}{\gamma}\right)^n, \quad t\ge 0.$$

We choose $h>0$ such that

$$\frac{2\overline{M}M\alpha(h)}{\gamma} < \frac{1}{2}.$$

Consequently, the limit $u := \lim_{n\to+\infty} u^{(n)}$ exists uniformly on $[-r,+\infty)$ and u is continuous on $[-r,+\infty)$. Moreover, $|u_t| < h$ for $t\ge 0$ and $u_0^S = \varphi$. In order to prove that u is a solution of Eq.(7.8), we introduce the following notation

$$\begin{aligned} v(t) = \ &\left| u_t - U(t)\varphi - \lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^S d\tau \right. \\ &\left. - \lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^{US} d\tau \right|. \end{aligned}$$

We obtain

$$\begin{aligned} v(t) \le \ &\left|u_t - u_t^{(n+1)}\right| \\ &+\left|\lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^S - \left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau^{(n)})\right)^S\right) d\tau\right| \\ &+\left|\lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau)\right)^{US} - \left(\widetilde{B_\lambda}X_0 g(\tau,u_\tau^{(n)})\right)^{US}\right) d\tau\right|. \end{aligned}$$

Moreover, we have

$$u_t - u_t^{(n+1)} = \sum_{k=n+1}^{+\infty}\left(u_t^{(k+1)} - u_t^{(k)}\right).$$

It follows that

$$v(t) \le 2h\left[1 + \frac{2\overline{M}M\alpha(h)}{\gamma}\right]\sum_{k=n+1}^{+\infty}\left(\frac{2\overline{M}M\alpha(h)}{\gamma}\right)^k.$$

Consequently, $v = 0$ on $[0, +\infty)$. To show the uniqueness suppose that w is also a solution of Eq.(7.8) with $|w_t| < h$ for $t \geq 0$. Then,

$$\left|w_t - u_t^{(n+1)}\right| \leq \left|\lim_{\lambda\to+\infty}\int_0^t U(t-\tau)\left(\left(\widetilde{B_\lambda}X_0 g(\tau, w_\tau)\right)^S - \left(\widetilde{B_\lambda}X_0 g(\tau, u_\tau^{(n)})\right)^S\right) d\tau\right|$$
$$+\left|\lim_{s\to+\infty}\lim_{\lambda\to+\infty}\int_s^t U(t-\tau)\left(\left(\widetilde{B_\lambda}X_0 g(\tau, w_\tau)\right)^{US} - \left(\widetilde{B_\lambda}X_0 g(\tau, u_\tau^{(n)})\right)^{US}\right) d\tau\right|.$$

This implies

$$\left|w_t - u_t^{(n+1)}\right| \leq 2h\left(\frac{2\overline{M}M\alpha(h)}{\gamma}\right)^n \to 0, \quad \text{as } n \to +\infty.$$

This proves the uniqueness and completes the proof.

8. **Periodic and almost periodic solutions**

In this section, we are concerned with the existence of periodic and almost periodic solutions of Eq.(6.1). As a consequence of the hyperbolicity, we obtain the following result.

Corollary 8.1. *Assume that* (**H3**) *holds and the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic. If the function* f *is* $\omega-$*periodic, then the only bounded solution of Eq.(6.1) given by Theorem 7.3 is also* $\omega-$*periodic.*

Proof. Let u be the unique bounded solution of Eq.(6.1). The function $u(.+\omega)$ is also a bounded solution of Eq.(6.1). The uniqueness property implies that $u = u(.+\omega)$.
We are concerned now with the existence of almost periodic solution of Eq.(6.1). Let $B(\mathbb{R}, E)$ be the space of bounded continuous function from $\mathbb{R}$ to E provided with the uniform norm topology.

Definition 8.2. [29] A function $h \in B(\mathbb{R}, E)$ is said to be almost periodic if and only if the set

$$\{h_\sigma : \sigma \in \mathbb{R}\},$$

where g_σ is defined by $h_\sigma(t) = h(t+\sigma)$, for $t \in \mathbb{R}$, is relatively compact in $B(\mathbb{R}, E)$.

Theorem 8.3. *Assume that* (**H3**) *holds and the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic. If the function* f *is almost periodic, then the only bounded solution of Eq.(6.1) is also almost periodic.*

Proof. Define the operator Q by

$$(Qf)(t) = \left(\lim_{\lambda\to+\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S d\tau\right)(0)$$
$$+\left(\lim_{\lambda\to+\infty}\int_{+\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau\right)(0), \text{ for } t \in \mathbb{R}.$$

Then, Q is a bounded linear operator from $B(\mathbb{R}, E)$ into $B(\mathbb{R}, E)$. By a sample computation, we obtain

$$(Qf)_\sigma = Q(f_\sigma), \text{ for } \sigma \in \mathbb{R}.$$

By the continuity of the operator Q, we deduce that $Q(\{f_\sigma : \sigma \in \mathbb{R}\})$ is relatively compact in $B(\mathbb{R}, E)$. This implies that if the function f is almost periodic, then the only bounded solution of Eq.(6.1) is also almost periodic.

We are concerned now with the existence of almost periodic solutions of Equation (7.8). We assume that

(**H5**) g is almost periodic in t uniformly in any compact set of $\mathcal{C}_E$. This means that for each $\varepsilon > 0$ and any compact set K of $\mathcal{C}_E$ there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property that

$$\sup_{t\in\mathbb{R},\varphi\in K} |g(t+\tau,\varphi)-g(t,\varphi)| < \varepsilon.$$

It's well known that if the function g is almost periodic in t uniformly in any compact set of $\mathcal{C}_E$ and if v is an almost periodic function, then the function $t \to g(t, v_t)$ is also almost periodic.

(**H6**) $|g(t,\varphi_1)-g(t,\varphi_2)| \leq k_1 |\varphi_1-\varphi_2|,\ t\in\mathbb{R}$ and $\varphi_1, \varphi_2 \in X$.

Proposition 8.4. *Assume that* (**H3**), (**H5**) *and* (**H6**) *hold. If the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic, then for* k_1 *small enough, Eq.(7.8) has a unique almost periodic solution.*

Proof. Let $AP(\mathbb{R},E)$ be the Banach space of almost periodic functions from $\mathbb{R}$ to E endowed with the uniform norm topology. Consider the operator H defined on $AP(\mathbb{R},E)$ by

$$Hv = u,$$

where u is the unique almost periodic solution of Eq.(6.1) with $f = g(.,v_.)$. By a sample computation, we can see that there exists a positive constant K_2 such that

$$|Hv_1 - Hv_2| \leq k_1k_2 |v_1-v_2|,\ v_1, v_2 \in AP(\mathbb{R},E).$$

If k_1 is chosen such that $k_1k_2 < 1$, then the map H is a strict contraction in $AP(\mathbb{R},E)$. We deduce that H has a unique fixed point in $AP(\mathbb{R},E)$, which gives an almost periodic solution of Eq.(7.8).

9. **Application**

To illustrate the above results, we consider the following partial functional differential equations with diffusion which describes the evolution of a single diffusive animal species with population density u. For more details, about this model, we refer to [47].

$$(9.1)\quad \begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = a\dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + bw(t,\xi) + c\displaystyle\int_{-r}^{0} G(\theta)w(t+\theta,\xi)d\theta + f(w(t-r,\xi)), \\ \qquad\qquad\qquad\qquad t\geq 0,\ 0\leq\xi\leq\pi, \\ w(t,0) = w(t,\pi) = 0,\ t\geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi),\ -r\leq\theta\leq 0,\ 0\leq\xi\leq\pi. \end{cases}$$

where a, b, c and r are positive constants, $f : I\!R \to I\!R$ is a continuous function, $G : [-r,0] \to I\!R$ continuous and $w_0 : [-r,0]\times[0,\pi] \to I\!R$ is a continuous function. In order to rewrite Eq.(9.1) in the abstract equation (1.1), we introduce $E = C([0,\pi]; I\!R)$, the space of continuous function from $[0,\pi]$ to $I\!R$, provided with the uniform norm topology and we the linear operator $A : D(A) \subseteq E \to E$ by

$$\begin{cases} D(A) = \{y \in C^2([0,\pi]; I\!R) : y(0) = y(\pi) = 0\}, \\ Ay = ay''. \end{cases}$$

It is well-known, see [19], that

$$\begin{cases} (0,+\infty) \subset \rho(A), \\ \left|(\lambda I - A)^{-1}\right| \leq \dfrac{1}{\lambda}, \text{ for } \lambda > 0. \end{cases}$$

This implies that Assumption $(\mathbf{H}_1)$ is satisfied. On the other hand, we can see that

$$\overline{D(A)} = \{y \in E : y(0) = y(\pi) = 0\} \neq E.$$

Set

$$\begin{cases} x(t)(\xi) = w(t,\xi), t \geq 0, \ \xi \in [0,\pi], \\ \varphi(\theta)(\xi) = w_0(\theta,\xi), \theta \leq 0, \ \xi \in [0,\pi], \\ F(\phi)(\xi) = b\phi(0)(\xi) + f(\phi(-r)(\xi)) + c\displaystyle\int_{-r}^{0} G(\theta)\phi(\theta)(\xi)d\theta, \ \xi \in [0,\pi], \ \phi \in \mathcal{C}_E. \end{cases}$$

Then, Eq.(9.1) takes the following abstract form,

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + F(x_t), \ t \geq 0, \\ x_0 = \varphi \in \mathcal{C}_E. \end{cases}$$

We suppose that,

(**i**) f is locally Lipschitz continuous.

It follow that F is locally Lipschitz continuous. Let $\varphi \in \mathcal{C}_E$ such that $\varphi(0) \in \overline{D(A)}$. Then, Theorem 2.10 ensures the existence of a maximal interval of existence $[-r, b_{w_0})$ and a unique integral solution $w(t,\xi)$ on $[-r, b_{w_0}) \times [0,\pi]$.
To investigate that the integral solution w of Eq.(9.1) is a strict one, we add the following assumptions

(**ii**) f is continuously differentiable and f' is locally Lipschitz continuous,
(**iii**) $w_0 \in C^2([-r,0] \times [0,\pi]; E)$, with $\dfrac{\partial}{\partial\theta}w_0(0,0) = \dfrac{\partial}{\partial\theta}w_0(0,\pi) = 0$ and

$$\begin{aligned} \frac{\partial}{\partial\theta}w_0(0,\xi) &= a\frac{\partial^2}{\partial\xi^2}w_0(0,\xi) + bw_0(0,\xi)) \\ &+c\int_{-r}^{0} G(\theta)w_0(\theta,\xi))d\theta + f(w_0(-r,\xi)), \text{ for } \xi \in [0,\pi]. \end{aligned}$$

Then, F is continuously differentiable on $\mathcal{C}_E$ and for $\phi, \psi \in \mathcal{C}_E$, $\xi \in [0,\pi]$, we have

$$F'(\phi)(\psi)(\xi) = b\psi(0)(\xi) + c\int_{-\infty}^{0} G(\theta)\psi(\theta)(\xi)d\theta + f'(\phi(-r)(\xi))\psi(-r)(\xi).$$

F' is also locally Lipschitz continuous on $\mathcal{C}_E$. Consequently, all the conditions in Theorem 5.2 are satisfied. Hence, w is a strict solution of Eq. (9.1).
In order to study the stability, we assume that,

(**iv**) f is continuously differentiable at 0, $f(0) = 0$, $f'(0) = 0$ and f is globally Lipschitz.

Then, F is continuously differentiable at 0 with $F(0) = 0$ and F is globally Lipschitz on $\mathcal{C}_E$. Consider the linearized equation of (9.1) corresponding to the derivative $F'(0)$ at 0,

$$(9.2)\quad \begin{cases} \dfrac{\partial}{\partial t}w(t,\xi) = a\dfrac{\partial^2}{\partial \xi^2}w(t,\xi) + bw(t,\xi) + c\displaystyle\int_{-r}^{0} G(\theta)w(t+\theta,\xi)d\theta, t \geq 0,\ 0 \leq \xi \leq \pi \\ w(t,0) = w(t,\pi) = 0,\ \ t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi),\ -r \leq \theta \leq 0,\ 0 \leq \xi \leq \pi. \end{cases}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$ given by

$$\begin{cases} D(A_0) = \{y \in C^2([0,\pi]; I\!R) : y(0) = y''(0) = y(\pi) = y''(\pi) = 0\}, \\ A_0 y = ay''. \end{cases}$$

Then, A_0 generates a strongly continuous semigroup on $\overline{D(A)}$ which is compact. Let A_U be the infinitesimal generator of the solution semigroup associated to Eq.(9.2) and $\sigma_p(A_U)$ denote the point spectrum of A_U. Then, $\lambda \in \sigma_p(A_U)$ if and only if there exists $\phi \in D(A_U)$, $\phi \neq 0$ such that $A_T\phi = \lambda\phi$. It follows that $\phi(\theta) = e^{\lambda\theta}y$ with $y \neq 0$, $y \in D(A)$ and $\lambda y = Ay + F'(0)(e^{\lambda \cdot}y)$. It follows that $\lambda \in \sigma_p(A_U)$ if and only if there exists $y \in D(A)$ and $y \neq 0$ such that

$$\lambda y = Ay + by + c\left(\int_{-r}^{0} G(\theta)e^{\lambda\theta}d\theta\right)y.$$

This means that

$$\lambda - b - c\int_{-r}^{0} G(\theta)e^{\lambda\theta}d\theta \in \sigma_p(A) = \sigma_p(A_0).$$

We know that the point spectrum $\sigma_p(A_0)$ of A_0 is given by

$$\sigma_p(A_0) = \left\{-an^2 : n \in I\!N^*\right\}.$$

Hence, the exponential stability of solutions of Eq.(9.2) is determined by the following characteristic equation

$$(9.3)\qquad \lambda - b - c\int_{-r}^{0} G(\theta)e^{\lambda\theta}d\theta = -an^2, \quad n \geq 1.$$

Lemma 9.1. *Assume that G is a positive function, $\int_{-r}^{0} G(\theta)d\theta = 1$ and $c < a - b$. Then, all roots of Eq.(9.3) have negative real part.*

Proof. Taking the real part in the characteristic Eq.(9.3), we get the following

$$\mathrm{Re}(\lambda) = b + c\int_{-r}^{0} G(\theta)e^{\mathrm{Re}(\lambda)\theta}\cos(\mathrm{Im}(\lambda\theta))d\theta - an^2,$$

which implies that

$$\mathrm{Re}(\lambda) \leq b + c\int_{-r}^{0} G(\theta)e^{\mathrm{Re}(\lambda)\theta}\cos(\mathrm{Im}(\lambda\theta))d\theta - a$$

and

$$\mathrm{Re}(\lambda) \leq b + c - a < 0.$$

Consequently, we deduce that all characteristic values have negative real part, which implies that the solution semigroup associated to Eq.(9.2) is exponentially stable.

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