Vortex Analysis of the Ginzburg-Landau Model of Superconductivity

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Preliminary version

Introduction

These notes report on recent mathematical work [33, 34, 35, 36, 37] which aims at describing minimizers of the Ginzburg-Landau functional in the presence of an applied magnetic field in terms of *vortices*. For some part these results were already known to be true by physicists and applied mathematicians, but were only recently rigourously proved. Also the mathematical approach has made the knowledge more accurate, and has clarified the validity regime of certain formal calculations.

0.1 History

We follow the lines of the introductory chapter of Tinkham's book [43].

The phenomenon of superconductivity was discovered in 1911 by K.Onnes, a dutch physicist who had succeeded in liquefying helium three years before. Performing low-temperature experiments with this new tool he observed that ordinary metals completely lose electrical resistance at temperatures of a few degrees Kelvin.

In the 1930's the Meissner effect, i.e. the expulsion of the magnetic field from a conductor when the temperature is lowered enough for it to become superconducting was discovered. Soonafter the London brothers proposed a modification of Maxwell's equations including the superconducting current. Even though this did not explain or predict how the superconducting current came about, it did account for the Meissner effect.

In 1950 the Ginzburg-Landau model — sometimes described as semi-phenomenological — was introduced. The London equation could be deduced from it but the structure of the Ginzburg-Landau model was much richer. In 1957 the Bardeen-Cooper-Schrieffer (BCS) theory gave an explanation of superconductivity in terms of the laws of quantum mechanics. The Ginzburg-Landau model could be derived from the BCS theory (Gorkov, 1959).

The Ginzburg-Landau model was studied by Abrikosov who, in a 1957 paper investigated the possibility of a new type of superconductors — the so-called type II superconductors — which could exist in a *mixed* state which was observed experimentally ten years later and proved very important for applications. Our aim is to derive some of Abrikosov's results in a mathematically rigorous way from the minimization of the Ginzburg-Landau energy.

The work of Onnes, Bardeen, Cooper, Schrieffer, Ginzburg, Landau and Abrikosov was distinguished by four Nobel Prizes, the latest of which was the 2003 prize for Ginzburg, Abrikosov and Legett.

0.2 The Ginzburg-Landau model

Consider a domain Ω in $\mathbb{R}^3$. The energy of a superconductor occupying Ω in the presence of a constant applied field He when the exterior region is insulating is written in the Ginzburg-Landau model

$$G(u, A) = G_0 + \int_{\mathbb{R}^3} \frac{|\operatorname{curl} A - \mathrm{He}|^2}{8\pi} + \int_{\Omega} \frac{1}{2m^*} \left| \left(\hbar\nabla - \frac{ie^*}{c} A \right) u \right|^2 + \alpha |u|^2 + \beta |u|^4. \tag{1}$$

In this expression $u : \Omega \to \mathbb{C}$ is the order parameter whose physical meaning is that of a condensed wave function for superconducting electron pairs and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the electromagnetic vector potential. Besides the constants $\hbar$ and c, additional constants m^* and e^* are present (see [43] for an explanation of these constants) as well as two quantities α and β that depend on the temperature T and on the superconducting material. Near a critical temperature T_c, it is assumed that β is constant and α is proportional to $T - T_c$ and has the same sign.

0.3 Adimensionalizing

The usual changes of variable make (1) more pleasant

$$\tilde{u}(x) = \sqrt{\frac{\beta}{|\alpha|}}u(\lambda x), \qquad \tilde{A}(x) = \frac{e^*}{\hbar c}\lambda A(\lambda x), \qquad \widetilde{\mathrm{He}} = \frac{e^*}{\hbar c}\mathrm{He}, \tag{2}$$

where λ is the *penetration depth* defined by

$$\lambda = \sqrt{\frac{\beta m^* c}{4\pi|\alpha|{e^*}^2}}.$$

The free energy then takes the form, letting $\varepsilon = 1/\kappa$,

$$J(\tilde{u}, \tilde{A}) = \widetilde{G}_0 + C\left(\frac{1}{2}\int\limits_{\mathbb{R}^3}\left|\mathrm{curl}\,\tilde{A} - \widetilde{\mathrm{He}}\right|^2 + \frac{1}{2}\int\limits_{\Omega}\left|\left(\nabla - i\tilde{A}\right)\tilde{u}\right|^2 + \frac{1}{2\varepsilon^2}\left(1 \pm |\tilde{u}|^2\right)^2\right) \tag{3}$$

where $\kappa = 1/\varepsilon = \lambda/\hbar\sqrt{m^*|\alpha|}$ is the Ginzburg-Landau parameter which depends on the material and varies little with temperature. The sign in $\left(1 \pm |u|^2\right)^2$ is the sign of the parameter α, i.e. is $+1$ if $T > T_c$ and -1 if $T < T_c$. In the first case the functional clearly has a unique critical point, namely $\tilde{u} \equiv 0$ and $\tilde{A}$ such that $\mathrm{curl}\,\tilde{A} \equiv \widetilde{\mathrm{He}}$. We are interested in the second case, where the phenomenon of superconductivity appears.

From now on we take $T < T_c$, assume the rescaling (2) and write u, A, He instead of $\tilde{u}, \tilde{A}, \widetilde{\mathrm{He}}$ for the rescaled quantities. In this scaling the length unit is the penetration depth. The object of our study is therefore

$$J(u, A) = \frac{1}{2}\int\limits_{\mathbb{R}^3}|\mathrm{curl}\,A - \mathrm{He}|^2 + \frac{1}{2}\int\limits_{\Omega}|(\nabla - iA)\,u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2. \tag{4}$$

Here $(\nabla - iA)\,u$ is the complex vector $(\partial_1^A u, \partial_2^A u, \partial_3^A u)$, where $\partial_k^A u = \partial_k u - iA_k u$.

0.4 Dimension reduction

A natural special case is that where the domain is an infinite cylinder in $\mathbb{R}^3$ and He is parallel to the axis (think of an infinitely long insulated wire). Assuming translational invariance of (u, A) and invariance with respect to reflections across a plane perpendicular to the axis we have, taking the third coordinate axis as the cylinder's axis

$$\mathrm{He} = h_{\mathrm{ext}}(0, 0, 1), \quad u(x, y, z) = u(x, y), \quad A(x, y, z) = (A_1(x, y), A_2(x, y), 0).$$

Then the Ginzburg-Landau energy per unit length of (u, A) is

$$J(u, A) = \frac{1}{2}\int\limits_{\mathbb{R}^2}|\mathrm{curl}\,A - h_{\mathrm{ext}}|^2 + \frac{1}{2}\int\limits_{\Omega}|(\nabla - iA)\,u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2, \tag{5}$$

Where $\Omega \subset \mathbb{R}^2$ is the cross section of the cylinder, $\mathrm{curl}\,A = \partial_1 A_2 - \partial_2 A_1$ and $h_{\mathrm{ext}} \geq 0$ is the intensity of the applied field. Our main goal will be to minimize this functional and describe its minimizers for different values of $\varepsilon, h_{\mathrm{ext}}$.

0.5 Notation

Given two complex numbers z, w we let $(z,w) = 1/2(\bar{z}w + z\bar{w})$, which is the inner product of z and w seen as vectors in $\mathbb{R}^2$.

Partial derivatives are written $\partial_1 u$, $\partial_2 u$,...

for any smooth bounded domain in $\mathbb{R}^2$ and any $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ we let

$$F_\varepsilon(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2, \tag{6}$$

$$J_\varepsilon(u,A,\Omega) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + (h-h_{\text{ext}})^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2,$$

where $h_{\text{ext}} > 0$ is the applied magnetic field, and

$$\nabla_A u = \nabla u - iAu, \quad h = \operatorname{curl} A := \partial_1 A_2 - \partial_2 A_1.$$

$(iu, \nabla_A u)$ denotes the vector with real components $(iu, \partial_1 u - iA_1 u), (iu, \partial_2 u - iA_2 u)$.

We denote by $|\Omega|$ the two-dimensional Lebesgue measure of Ω

H_0^1 denotes the closure of smooth functions with compact support in Ω in the H^1 norm $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$. Similarly $W_0^{1,p}$ denotes the closure of smooth functions with compact support in Ω in the $W^{1,p}$ norm while $W_0^{-1,p}$ denotes the dual of $W^{1,q}$, where $1/p + 1/q = 1$.

Chapter 1

Minimization of the Ginzburg-Landau Functional

We now show that the functional J can indeed be minimized. The main difficulty is the *gauge invariance* of the problem.

1.1 Gauge invariance

Definition 1. *For any (smooth) function $f : \mathbb{R}^2 \to \mathbb{R}$, any $u : \Omega \to \mathbb{C}$ and any $A : \mathbb{R}^2 \to \mathbb{R}^2$ we define*

$$v = ue^{if}, \quad B = A + \nabla f$$

and we say the configuration (v, B) is gauge equivalent *to (u, A). The transformation from (u, A) to (v, B) is called a* gauge transformation.

Then we have

Proposition 1. *If (v, B) is gauge equivalent to (u, A) then $J(v, B) = J(u, A)$.*

Proof. if $v = ue^{if}$ and $B = A+\nabla f$ for some real-valued function f, then $\operatorname{curl} B = \operatorname{curl} A$, $|v| = |u|$ and

$$\nabla v = (\nabla u + iu\nabla f)\, e^{if}, \quad iBu = (iAu + iu\nabla f)\, e^{if}$$

hence $(\nabla - iA)\, u = e^{if}\, (\nabla - iA)\, v$. Replacing in (5) proves the proposition. $\square$

Remark 1. *This invariance by a large group of transformation (all smooth real-valued functions) poses a problem for the minimization of J. Indeed if $\{(u_n, A_n)\}_n$ is a minimizing sequence then for any sequence of functions $\{f_n\}_n$, we have that $\{(u_n e^{if_n}, A_n + \nabla f_n)\}_n$ is also minimizing, however wild the functions f_n are. Thus no good bounds on $\{(u_n, A_n)\}_n$ can be deduced from the fact that $J(u_n, A_n)$ is bounded independently of n. The Coulomb Gauge will solve this problem.*

1.2 The Coulomb Gauge

Definition 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. We say $A : \Omega \to \mathbb{R}^2$ satisfies the Coulomb gauge condition in Ω if*

$$\begin{cases} div\, A = 0 & in\ \Omega \\ A.\nu = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the outward pointing normal to $\partial\Omega$.

We have

Proposition 2. *For any smooth bounded domain $\Omega \subset \mathbb{R}^2$ and $A \in H^1(\Omega, \mathbb{R}^2)$, there exists a gauge transformation $f \in H^2(\Omega)$ such that $B = A + \nabla f$ satisfies the Coulomb gauge condition in Ω.*

Proof. Let f solve

$$\begin{cases} \Delta f = -\operatorname{div} A & \text{in } \Omega \\ \partial_\nu f = -A.\nu & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This is possible since $\int_\Omega \operatorname{div} A = \int_{\partial\Omega} A.\nu$ and the solution is unique modulo a constant. Then $A + \nabla f$ satisfies the desired conditions. □

The following estimate is crucial for the minimization of (5)

Proposition 3. *Let Ω be a smooth, bounded, simply connected domain in $\mathbb{R}^2$. There exists a constant $C > 0$ such that if $A : \Omega \to \mathbb{R}^2$ satisfies the Coulomb gauge condition then*

$$\|A\|^2_{H^1(\Omega,\mathbb{R}^2)} \le C \|\operatorname{curl} A\|^2_{L^2(\Omega)}$$

Proof. Since Ω is simply connected and $\operatorname{div} A = 0$ in Ω there exists a function f such that $A = (-\partial_2 f, \partial_1 f)$. Then $A.\nu = 0$ on $\partial\Omega$ implies that f is constant on $\partial\Omega$ and subtracting the constant we may assume $f = 0$ on $\partial\Omega$. Moreover $\operatorname{curl} A = \Delta f$. Standard elliptic theory then implies that $\|f\|^2_{H^2(\Omega)} \le C\|\operatorname{curl} A\|^2_{L^2(\Omega)}$, from which the result follows. □

1.3 Minimization of the Ginzburg-Landau Energy

From now on Ω is a smooth bounded simply connected domain in $\mathbb{R}^2$.

1.3.1 Restriction to Ω

The natural space for the minimization of (5) is

$$X = \{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \mid (\operatorname{curl} A - h_{\text{ext}}) \in L^2(\mathbb{R}^2)\}. \tag{1.3}$$

To avoid the technical difficulties of minimizing (5) over X, we use the following trick. Let

$$J_\Omega(u, A) = \frac{1}{2} \int_\Omega |\operatorname{curl} A - h_{\text{ext}}|^2 + \frac{1}{2} \int_\Omega |(\nabla - iA)\, u|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2, \tag{1.4}$$

Defined over

$$X_\Omega = \{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)\}. \tag{1.5}$$

It is clear that if $(u, A) \in X$ then its restriction to Ω is in X_Ω and

$$J_\Omega(u, A) \le J(u, A). \tag{1.6}$$

Reciprocally we have

Lemma 1. *Let $(u, A) \in X_\Omega$. Then A can be extended to $\mathbb{R}^2$ in a way such that $J(u, A) = J_\Omega(u, A)$*

Proof. There exists $B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{curl} B = \operatorname{curl} A$ in Ω and $\operatorname{curl} B = h_{\text{ext}}$ outside Ω. Then, since Ω is simply connected, there exists a function $f : \Omega \to \mathbb{R}$ such that $B = A + \nabla f$ in Ω. It follows that $J_\Omega(u, A) = J_\Omega(ue^{if}, B)$ and since $\operatorname{curl} B = h_{\text{ext}}$ outside Ω, we find $J_\Omega(u, A) = J(ue^{if}, B)$. By extending f to $\mathbb{R}^2$ in an arbitrary way and gauge transforming (ue^{if}, B) by $-f$, the Lemma is proved. □

This Lemma together with (1.6) proves

Proposition 4. *The minimum of J over X is equal to the minimum of J_Ω over X_Ω. Moreover minimizers of J restrict to minimizers of J_Ω and reciprocally minimizers of J_Ω can be extended to minimizers of J.*

We prove below that a minimizer of J_Ω hence a minimizer of J, exists.

1.3.2 Minimization of J

Proposition 5. *The minimum of J over X is achieved.*

Proof. From Proposition 4 it suffices to prove that the minimum of J_Ω over X_Ω is achieved. Let $\{(u_n, A_n)\}_n$ be a minimizing sequence for J_Ω. We may assume by density that elements of the sequence are smooth. Also, using Proposition 2 we may assume A_n satisfies the Coulomb gauge condition in Ω for all n. Using the bound $J_\Omega(u_n, A_n) \le C$, where C is independent of n, we find that $\|1 - |u_n|^2\|_{L^2}$, $\|(\nabla - iA_n)u_n\|_{L^2}$ and $\|\operatorname{curl} A_n - h_{\text{ext}}\|_{L^2}$ are bounded independently of n. Therefore $\{\operatorname{curl} A_n\}_n$ is bounded in L^2 and thus , from Proposition 3, $\{A_n\}_n$ is bounded in H^1.

Let $\nabla u_n = (\nabla - iA_n)u_n + iA_n u_n$. Since $\{A_n\}_n$ is bounded in H^1 it is bounded in every L^q by Sobolev embedding. Because $\{u_n\}_n$ is bounded in L^4 we find that $\{iA_n u_n\}_n$ is bounded in $L^{4-\varepsilon}$ for any $\varepsilon > 0$ and in particular in L^2. Thus $\{\nabla u_n\}_n$ is bounded in L^2 and $\{u_n\}_n$ is bounded in H^1.

We may then extract a subsequence such that $\{u_n\}_n$ and $\{A_n\}_n$ converges to some (u_0, A_0) weakly in H^1 and, by compact Sobolev embedding, strongly in every L^q. We now show that (u_0, A_0) is a minimizer of J_Ω.

By strong L^4 convergence $\liminf_n \|1 - |u_n|^2\|_{L^2}^2 = \|1 - |u_0|^2\|_{L^2}$. Also, $\|\operatorname{curl} A - h_{\text{ext}}\|_{L^2}^2$ is a convex function of A which is continuous in the H^1 norm hence it is weakly lower semicontinuous in H^1. Therefore $\liminf_n \|\operatorname{curl} A - h_{\text{ext}}\|_{L^2}^2 \ge \|\operatorname{curl} A_0 - h_{\text{ext}}\|_{L^2}^2$. It remains to check that $\liminf_n \|(\nabla - iA_n)u_n\|_{L^2}^2 \ge \|(\nabla - iA_0)u_0\|_{L^2}^2$. This is left as an exercise, noting that

$$|(\nabla - iA)u|^2 = |\nabla u|^2 - 2\,(iAu, (\nabla - iA)u) + |A|^2|u|^2.$$

□

1.4 Euler-Lagrange equations

Definition 3. *We say $(u, A) \in X$ is a critical point of J if for every (v, B) smooth and compactly supported we have*

$$\frac{d}{dt} J(u + tv, A + tv)_{|t=0} = 0.$$

Clearly a minimizer of J is a critical point.

Proposition 6. *If $(u, A) \in X$ is a critical point of J then it satisfies*

$$\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2} u(1 - |u|^2) \quad \text{in } \Omega \\ -\nabla^\perp h = (iu, \nabla_A u) \quad \text{in } \Omega \\ h = h_{\text{ext}} \quad \text{in } \mathbb{R}^2 \setminus \Omega \\ \nu.\nabla_A u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Where we have set $h = \operatorname{curl} A$ and used the following notation. The *covariant laplacian* is defined by

$$(\nabla_A)^2 u = \partial_1^A(\partial_1^A u) + \partial_2^A(\partial_2^A u), \tag{1.8}$$

the *covariant gradient* is

$$\nabla_A u = (\nabla - iA)u$$

and the *current* is a vector in $\mathbb{R}^2$ defined by

$$(iu, \nabla_A u) = \left((iu, \partial_1^A u), (iu, \partial_2^A u)\right), \tag{1.9}$$

where, for complex numbers $w = x + iy$, $w = x' + iy'$, we let $(z, w) = xx' + yy'$. Finally,

$$\nu.\nabla_A u = \nu^1 \partial_1^A u + \nu^2 \partial_2^A u.$$

The derivation of (1.7) is made very close to, say, the derivation of the Laplace equation from minimization of the Dirichlet energy by using the following lemma, the proof of which is left as an exercise.

Lemma 2. *For arbitrary complex valued functions u, v and any A,*

$$\partial_k(u,v) = (\partial_k^A u, v) + (u, \partial_k^A v).$$

Proof of the Proposition. We have

$$\frac{d}{dt}J(u+tv, A+tv)_{|t=0} = \int_\Omega (\nabla_A u, \nabla_A v) + (\nabla_A u, -iBu) - \frac{(u,v)}{\varepsilon^2}(1-|u|^2) + \int_{\mathbb{R}^2} (\operatorname{curl} A - h_{\text{ext}}) \operatorname{curl} B,$$

where $(\nabla_A u, \nabla_A v) = (\partial_1^A u, \partial_1^A v) + (\partial_2^A u, \partial_2^A v)$. Using the Lemma we have

$$(\nabla_A u, \nabla_A v) = \sum_{h=1}^2 \partial_k(\partial_k^A u, v) - ((\partial_k^A)^2 u, v) = \operatorname{div}(\nabla_A u, v) - ((\nabla_A)^2 u, v),$$

where $(\nabla_A u, v) = \big((\partial_1^A u, v), (\partial_1^A u, v)\big)$. Therefore, integrating by parts

$$\frac{d}{dt}J(u+tv, A+tv)_{|t=0} = \int_{\partial\Omega} (\nu.\nabla_A u, v) +$$
$$+ \int_\Omega -\big((\nabla_A)^2 u, v\big) - (iu, \nabla_A u)\cdot B - \frac{(u,v)}{\varepsilon^2}(1-|u|^2) - \int_{\mathbb{R}^2} \nabla^\perp(\operatorname{curl} A - h_{\text{ext}}).B.$$

Since this is true for any (v, B) we find $-\nabla^\perp(h - h_{\text{ext}}) = (iu, \nabla_A u)$ and

$$-(\nabla_A)^2 u = \frac{u}{\varepsilon^2}(1-|u|^2)$$

in Ω, while $\nabla^\perp(h - h_{\text{ext}}) = 0$ outside Ω. Since h_{ext} is constant h is constant outside Ω and this constant must be h_{ext} since the configuration has finite energy. The boundary conditions follow as well. □

1.4.1 Properties of Critical Points

Proposition 7 (regularity). *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If (u, A) is a critical point of J and if A satisfies the Coulomb gauge condition* (1.1), *then u, A are smooth in Ω and A is smooth outside Ω.*

Proof. Together with the Coulomb gauge condition, the Ginzburg-Landau equations (1.7) become

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) - 2i(A.\nabla)u - |A|^2 u \quad \text{in } \Omega \\ -\Delta A = (iu, \nabla u - iAu) \quad \text{in } \Omega \\ \quad h = h_{\text{ext}} \quad \text{outside } \Omega \\ \nu.\nabla_A u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

The first equation is obtained by expanding $(\nabla_A)^2 u$. To obtain the second equation from (1.7), note that

$$-\nabla^\perp h = (\partial_2(\partial_1 A_2 - \partial_2 A_1), -\partial_1(\partial_1 A_2 - \partial_2 A_1)). \tag{1.11}$$

Differentiating $\partial_1 A_1 + \partial_2 A_2 = 0$ with respect to both variables we find $\partial_{12} A_2 = -\partial_{11} A_1$ and $\partial_{12} A_1 = -\partial_{22} A_2$. Replacing in (1.11) yields $-\nabla^\perp h = -\Delta A$ and thus (1.10).

But (1.10) are a couple of elliptic equation for which we easily derive regularity by bootstrapping arguments. Since (u, A) are both in H^1, hence in every L^q, the right-hand side of the equations (1.7) are in L^p for any $p < 2$ and therefore (u, A) are both in $W^{2,p}$ by standard elliptic theory, and therefore in every $W^{1,q}$, etc...

Note that the above argument yields interior regularity, boundary regularity requires a more careful inspection of the boundary conditions $h = h_{\text{ext}}$, $\nu.\nabla_A u = 0$ supplemented by $A.\nu = 0$. See [10] for more details. □

Proposition 8. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If (u, A) is acritical point of J then $|u| \leq 1$ in Ω.*

Proof. This is a consequence of the maximum principle. Taking the scalar product of the first equation in (1.10) with u we find

$$-(\Delta u, u) = \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2) - 2(i(A.\nabla)u, u) - |A|^2|u|^2.$$

Therefore

$$-\frac{1}{2}\Delta|u|^2 = -(\Delta u, u) - |\nabla u|^2 = \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2) - 2(i(A.\nabla)u, u) - |A|^2|u|^2 - |\nabla u|^2.$$

Noting that

$$|\nabla_A u|^2 = |\nabla u|^2 + 2(i(A.\nabla)u, u) + |A|^2|u|^2$$

we find

$$-\frac{1}{2}\Delta|u|^2 = \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2) - |\nabla_A u|^2.$$

and therefore

$$-\frac{1}{2}\Delta\left(1 - |u|^2\right) + \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2) = |\nabla_A u|^2.$$

We multiply this equation by $(1 - |u|^2)_- = min(1 - |u|^2, 0)$ and integrate in Ω to find

$$\frac{1}{2}\int_\Omega -(1 - |u|^2)_-\Delta\left(1 - |u|^2\right) + \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2)(1 - |u|^2)_- \leq 0.$$

Integrating by parts we get

$$-\frac{1}{2}\int_{\partial\Omega}(1 - |u|^2)_-\partial_\nu\left(1 - |u|^2\right) + \int_\Omega \nabla(1 - |u|^2).\nabla(1 - |u|^2)_- + \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2)(1 - |u|^2)_- \leq 0.$$

It is a well-known property of Sobolev functions that $\nabla f = 0$ a.e. on any level set $\{f = a\}$ therefore the above inequality may be rewritten

$$-\frac{1}{2}\int_{\partial\Omega}(1 - |u|^2)_-\partial_\nu\left(1 - |u|^2\right) + \int_{|u|\geq 1} |\nabla(1 - |u|^2)|^2 + \frac{1}{\varepsilon^2}|u|^2(1 - |u|^2)^2 \leq 0.$$

The Neumann boundary condition $\nu \cdot \nabla_A u = 0$ implies, taking the scalar product with u, that $\partial_\nu|u| = 0$ thus the above inequality implies, considering the smoothness of u in Ω, that $\{x \in \Omega \mid |u| > 1\} = \varnothing$. □

Chapter 2

Introduction to critical fields in $\mathbb{R}^2$

We now begin to adress the main topic of these lectures, namely to describe the minimizers of J when the parameters $\varepsilon, h_{\text{ext}}$ vary. We wish to draw a phase diagram, i.e. to tell what kind of minimizers are to be found in different areas of the $(\varepsilon, h_{\text{ext}})$ plane. These areas will be separated by what it usually called *critical lines*.

The local state of the material at a point x is described by $u(x)$, the so-called order parameter. But due tu gauge invariance, only $|u|$ has a physical significance. In the adimensionalized form of the functional we use, $|u| = 1$ means the material is superconducting and $|u| = 0$ means it is normal.

In this chapter we do not aim at mathematical rigor. Also, we take the domain Ω to be $\mathbb{R}^2$, which corresponds to an infinetely large superconducting sample.

2.1 Pure states

If $\Omega = \mathbb{R}^2$, boundary conditions should be ignored and the system (1.7) reduces to the first two equations. We distinguish two solutions

The superconducting solution for which $u = 1$ is a constant and $A = 0$. It has infinite energy if $h_{\text{ext}} > 0$ but its energy per unit area is $h_{\text{ext}}^2/2$.
The normal solution. If $u = 0$ is a constant and $h = \operatorname{curl} A$ also then (u, A) is a solution. The energy per unit area is $(h - h_{\text{ext}})^2/2 + 1/4\varepsilon^2$, thus among these solutions the least energetic is the one for which $h = h_{\text{ext}}$. What we call the normal solution is $u = 0$ and A such that $\operatorname{curl} A = h_{\text{ext}}$. It is really *one* solution modulo gauge transformations. Its energy per unit area is $1/4\varepsilon^2$.

Therefore if we assume that the energy minimizer is one of these two solutions we find a first critical line

$$\boxed{H_c(\varepsilon) = \frac{1}{\varepsilon\sqrt{2}}}, \tag{2.1}$$

meaning that if for a given value of ε we have $h_{\text{ext}} < H_c(\varepsilon)$ then the superconducting solution is more favorable than the normal one, whereas if $h_{\text{ext}} > H_c(\varepsilon)$ it is the reverse.

2.2 Critical line H_{c2}

The normal solution satisfies $u = 0$ everywhere. Thus if we look for solutions near the normal solution, for instance bifurcating solutions, we may in first approximation linearize the equation in u, that is drop the term $u|u|^2$ in the right hand side of the first Ginzburg-Landau equation in (1.7). Abrikosov showed that solutions for this linearized equation exist if $\varepsilon < \sqrt{2}$ and h_{ext} is equal to the critical value

$$H_{c2}(\varepsilon) = \boxed{\frac{1}{\varepsilon^2}}. \tag{2.2}$$

Moreover he showed these solutions are more favorable energetically than the normal solution if h_{ext} decreases below this value. Recently Dutour [19] showed that solutions to the original (not linearized) equation existed that corresponded to the Abrikosov solutions.

The Abrikosov solutions are periodic, or rather are such that the gauge invariant quantities, such as $|u|$ and $h = \operatorname{curl} A$ are periodic. The zeroes of u form a lattice and around each zero u has a nonzero *winding number*. That is, writing $u = |u|e^{i\varphi}$, and working in polar coordinates (r, θ) centered at a zero of u, if $r > 0$ is small enough, the integer

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{\partial \varphi}{\partial \theta}(r, \theta)\, d\theta$$

is not zero. The points where u vanishes are called *vortices* and the integer above the *degree* of the vortex. At a vortex the induced magnetic field $h = \operatorname{curl} A$ has a local maximum.

Remark 2. *Note that when writing $u = |u|e^{i\varphi}$, the phase φ is not gauge invariant, however the degree of a vortex is.*

2.3 Critical line H_{c1}

Assume now $\varepsilon < \sqrt{2}$. If h_{ext} is high the normal solution is more favorable then the superconducting or Abrikosov solutions. Then, lowering h_{ext} below H_{c2}, the Abrikosov solutions become less energetic and the minimizer of the Ginzburg-Landau energy is supposedly one of them. The question is then to compute the critical value of h_{ext} below which the superconducting solution becomes in turn more favorable than the Abrikosov solutions. There is no reason for which this value should be given by (2.1), which was computed by comparing the normal and superconducting solutions. We call the new value H_{c1}, it should be smaller than H_c.

To simplify matters we will not compare the superconducting solution to an Abrikosov type solution, but rather to a single vortex solution, or rather approximate solution. The computations will be meaningful only when ε is small.

2.3.1 Approximate vortex

Our approximate solution will have — except for the pure states — the maximal symmetry allowed by the equations, i.e. rotational symmetry. We will look for (u, A) in the form

$$u(r, \theta) = f(r)e^{i\theta}, \quad A(r, \theta) = g(r)(-\sin\theta, \cos\theta). \tag{2.3}$$

Next we argue that if ε is small, then for $J(u, A)$ to be as small as possible $|u|$ should be close to 1 except on a small set. Moreover scaling arguments suggest that the area of this set should be of the order of ε^2. For this reason we let

$$f(r) = \begin{cases} r/\varepsilon & \text{if } r < \varepsilon \\ 1 & \text{otherwise.} \end{cases} \tag{2.4}$$

Now we need to define g in a reasonable way. Since the definition of u was rather arbitrary, or so it may seem, we will try to do a better job with A. The best would be of course to solve the Ginzburg-Landau equation for A, i.e.

$$-\nabla^{\perp} h = (iu, \nabla_A u),$$

where $h = \operatorname{curl} A$. If we write $u = \rho e^{i\varphi}$ — we will use the ansatz (2.3) in a while — then

$$\nabla u = \nabla\rho e^{i\varphi} + i\rho\nabla\varphi e^{i\varphi} - iA\rho e^{i\varphi}$$

therefore $(iu, \nabla_A u) = \rho^2(\nabla\varphi - A)$.

Thus when $\rho = 1$, the second Ginzburg-Landau equation becomes $-\nabla^\perp h = \nabla\varphi - A$, and taking the curl yields

$$-\Delta h + h = 0. \tag{2.5}$$

When ρ varies the equation for h is more complicated, but since this happens in a very small area we will account for it in a simplified way. We compute

$$-\int_{B(0,\varepsilon)} \Delta h = -\int_{\partial B(0,\varepsilon)} \nu\cdot\nabla h = \int_{\partial B(0,\varepsilon)} \tau\cdot\nabla^\perp h.$$

Assuming the second Ginzburg-Landau equation is satisfied together with (2.3), (2.4) we find

$$-\int_{B(0,\varepsilon)} \Delta h = \int_{\partial B(0,\varepsilon)} \tau\cdot(\nabla\varphi - A) = \int_{\partial B(0,\varepsilon)} \tau\cdot\nabla\theta - \int_{B(0,\varepsilon)} h.$$

Therefore

$$\int_{B(0,\varepsilon)} -\Delta h + h = 2\pi. \tag{2.6}$$

In view of (2.5), (2.6) which we recall are consequences of our ansatz together with the second Ginburg-Landau equation, we *define* h to be the positive solution to

$$-\Delta h + h = 2\pi\delta, \tag{2.7}$$

where δ is the Dirac mass at 0. The solution is a radial function in $\mathbb{R}^2$. We deduce A in the form (2.3) from the relation $h = \operatorname{curl} A$ by integrating it over the ball $B(0,r)$. This yields

$$\int_{\partial B(0,r)} A\cdot\tau = \int_{B(0,r)} h$$

and then, together with (2.7)

$$g(r) = \frac{1}{r} + h'(r). \tag{2.8}$$

2.3.2 The energy of the approximate vortex

We compute the energy of the configuration (u, A) defined by (2.3), (2.4), (2.7), (2.8). The energy in $\mathbb{R}^2$ is infinite, but we are really interested in the *difference* between the energy of (u, A) and that of the superconducting solution. Thus, writing B_r for $B(0, r)$, we let

$$\Delta(R) = J_{B_R}(u, A) - J_{B_R}(1, 0) = J_{B_R}(u, A) - \frac{1}{2}|B_R|h_{\text{ext}}^2, \tag{2.9}$$

and try to compute the limit of this quantity as R tends to $+\infty$. As in (1.4) we have used the notation $J_{B_R}(u, A)$ for the Ginzburg-Landau energy density integrated over the ball B_R. We split $\Delta(R)$ by writing $\Delta(R) = \alpha + \beta(R)$for any $R > \varepsilon$, where

$$\alpha = J_{B_\varepsilon}(u, A) - J_{B_\varepsilon}(1, 0), \quad \beta(R) = J_{B_R\setminus B_\varepsilon}(u, A) - J_{B_R\setminus B_\varepsilon}(1, 0). \tag{2.10}$$

To evaluate α and $\beta(R)$ we will need the following (see [43])

Lemma 3. *Let h be the positive solution to $-\Delta h + h = 2\pi\delta$. Then $h(r) = |\log r| + C + o(1)$ as $r \to 0$ and the corresponding behavior for the derivative also holds, i.e. $h'(r) = -1/r + o(1)$ as $r \to 0$. Moreover $h(r), h'(r) = O(e^{-r})$ as $r \to +\infty$.*

Now we can prove

Lemma 4. *Assuming $h_{ext} \le 1/\varepsilon^2$, there exists a constant C independent of $\varepsilon < 1$ such that $|\alpha| < C$.*

Proof. We let C denote a generic constant independent of $\varepsilon < 1$. From (2.3), (2.4) we have $|\nabla u| < C/\varepsilon$ in $\mathbb{R}^2$. From (2.3), (2.8) and Lemma 3 we find $|A| < C$ in B_1 and $\|h\|_{L^q} < C$ for any $q \ge 1$. Therefore, in B_ε,

$$|\nabla_A u|^2 \le \frac{C}{\varepsilon^2}, \quad \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \le \frac{C}{\varepsilon^2}$$

and since $-2h_{\text{ext}} \le (h-h_{\text{ext}})^2 - h_{\text{ext}}^2 \le h^2$,

$$\int_{B_\varepsilon} \left|(h-h_{\text{ext}})^2 - h_{\text{ext}}^2\right| \le C.$$

It follows that

$$|\alpha| = |J_{B_\varepsilon}(u,A) - J_{B_\varepsilon}(1,0)| = \left|J_{B_\varepsilon}(u,A) - \frac{h_{\text{ext}}^2}{2}|B_\varepsilon|\right| \le C.$$

□

Concerning $\beta(R)$ we have.

Lemma 5. *Let $\beta(h_{ext},\varepsilon) = \lim_{R\to+\infty}\beta(R)$. Then*

$$\beta(h_{ext},\varepsilon) = \pi\left(\log\varepsilon - 2h_{ext}\right)(1+o(1)) + O(1),$$

where $o(1)$ and $O(1)$ are meant as $\varepsilon \to 0$ and are independent of h_{ext}.

Proof. In $\mathbb{R}^2 \setminus B_\varepsilon$ we have $|u| = 1$. Therefore as noted above the second Ginzburg-Landau equation becomes $-\nabla^\perp h = \nabla\varphi - A$, thus

$$J_{B_R\setminus B_\varepsilon}(u,A) = \frac{1}{2}\int_{B_R\setminus B_\varepsilon} |\nabla h|^2 + (h-h_{\text{ext}})^2.$$

Therefore

$$\beta(R) = \frac{1}{2}\int_{B_R\setminus B_\varepsilon} |\nabla h|^2 + h^2 - 2hh_{\text{ext}}.$$

Integrating by parts and using (2.7) yields

$$\int_{B_R\setminus B_\varepsilon} |\nabla h|^2 + h^2 = \int_{\partial B_R} h(\nu\cdot\nabla h) - \int_{\partial B_\varepsilon} h(\nu\cdot\nabla h) = 2\pi R h(R)h'(R) - 2\pi\varepsilon h(\varepsilon)h'(\varepsilon).$$

Using (2.7) again,

$$\int_{B_R\setminus B_\varepsilon} h = \int_{B_R\setminus B_\varepsilon} \Delta h = 2\pi R h'(R) - 2\pi\varepsilon h(\varepsilon).$$

Therefore $\beta(R) = \pi R h'(R)(h(R) - 2h_{\text{ext}}) - \pi\varepsilon h'(\varepsilon)(h(\varepsilon) - 2h_{\text{ext}})$. From Lemma 3, $h'(R)$ goes to zero exponentially fast as $R \to +\infty$ and as $\varepsilon \to 0$ we have $h'(\varepsilon) = 1/\varepsilon + o(1)$, $h(\varepsilon) = |\log\varepsilon| + O(1)$. The Lemma follows. □

2.3.3 The critical line H_{c1}

In view of Lemmas 4, 5, We find that

$$\lim_{R \to +\infty} \Delta(R) = \pi \log \frac{1}{\varepsilon} - 2\pi h_{\mathrm{ext}} + C,$$

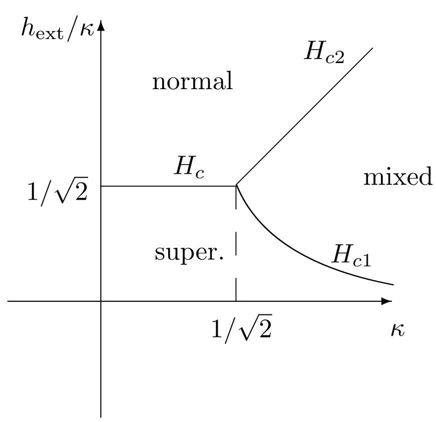
where C is bounded independently of ε. Clearly this result is meaningful only for small values of ε, but shows that in this case there exists a critical value

$$\boxed{H_{c1}(\varepsilon) \approx \tfrac{|\log \varepsilon|}{2}} \tag{2.11}$$

such that if h_{ext} is below $H_{c1}(\varepsilon)$, the superconducting solution is energetically favorable compared to the approximate vortex whereas it is the opposite if $h_{\mathrm{ext}} > H_{c1}(\varepsilon)$.

Several remarks can be made at this point. First, the equivalent for H_{c1} as $\varepsilon \to 0$ that we computed is not very sensitive to the way we construct the approximate vortex. We see from Lemma 4, for instance, that the contribution of B_ε is negligible when computing the value $|\log \varepsilon|/2$.

The second remark is that the approximate vortex is quite different from the Abrikosov solutions, shouldn't there be other critical values of h_{ext} marking the transition from one vortex to two vortex and so on ? The answer is that although the approximate vortex allows to compute the right critical value, the least energy configuration when h_{ext} crosses the line should look more like a vortex lattice similar to an Abrikosov solution. The reason for this is that if adding a vortex to the superconducting solution allows to gain somme energy, then adding many vortices allows to gain more energy. The minimizer will then be a lattice of vortex solutions glued together.



2.4 Phase diagram

We may sum up the previous analysis in the above diagram, where we have plotted the critical lines in the plane (x, y), where $x = \kappa = 1/\varepsilon$ is the Ginzburg-Landau parameter and $y = h_{\mathrm{ext}}/\kappa$. To the left of $\kappa = 1/\sqrt{2}$ the Abrikosov solutions do not exist and there is a single critical line separating the domains where the energy minimizer is respectively the normal and superconducting solution. When $\kappa > 1/\sqrt{2}$ the critical line H_c divides in two: the critical line $H_{c2}(\kappa) = \kappa^2$ above which the normal solution is the minimizer and the critical line $H_{c1}(\kappa)$ which behaves for large κ as $\log \kappa/2$ and below which the superconducting solution is the minimizer. In between these two lines we expect the minimizer to be an Abrikosov type solution, i.e. a lattice of vortices. Note that we have given no justification for the fact that the H_{c1} line meets H_c and H_{c2} at the same point, in fact we know of none. We re fer to the lectures by A.Aftalion for an account of numerical data that shows the complexity of the problem.

The rest of these lectures will be devoted, roughly speaking, to the rigorous derivation of (2.11) from the minimization of the Ginzburg-Landau functional. Before we attack this problem we review some mathematical tools which will be useful to us.

Chapter 3

Mathematical Preliminaries

3.1 Degree theory

3.1.1 Definition

Assume Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, then there is a natural orientation on $\partial\Omega$ given by the unit vector τ such that if ν is the *inward* pointing unit normal vector on $\partial\Omega$ then (τ, ν) is at each point of the boundary a direct othonormal frame. If $g : \partial\Omega \to \mathbb{S}^1$ is a sufficiently smooth map we can then define the *degree* of g by

$$\deg(g, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} (ig, \partial_\tau g)\, ds \tag{3.1}$$

Since g is smooth, it can be written locally as $g = \exp(i\varphi)$ for some smooth real-valued function φ, the imaginary part of a complex logarithm of g. Then the integrand in (3.1) is $\partial_\tau \varphi$, in particular the degree is an integer. With our choice of orientation $g(z) = z$ has degree one on the boundary of the unit disk.

The degree seen as a function defined on $C^\infty(\partial\Omega, \mathbb{S}^1)$ is continuous in the C^0 norm. Indeed if $g_0, g_1 \in C^\infty(\partial\Omega, \mathbb{S}^1)$ are such that $\sup_{x\in\partial\Omega} |g_0(x) - g_1(x)| < 2$, then for any $0 \le t \le 1$ the map

$$g_t(x) = \frac{tg_1(x) + (1-t)g_0(x)}{|tg_1(x) + (1-t)g_0(x)|}$$

is in $C^\infty(\partial\Omega, \mathbb{S}^1)$ and $\deg(g_t, \partial\Omega)$ is a continuous function of t (we leave it to the reader to check that the denominator does not vanish). But $\deg(g_t, \partial\Omega)$ is an integer and is therefore constant, hence $\deg(g_0, \partial\Omega) = \deg(g_1, \partial\Omega)$. This implies that the function $g \to \deg(g, \partial\Omega)$ can be continuously extended to the space of continuous $\mathbb{S}^1$-valued maps, which is the classical setting of degree theory (see [18]).

An alternative expression for the degree allows to define it for traces of Sobolev maps $u \in H^1(\Omega, \mathbb{C})$ such that $|u| = 1$ on $\partial\Omega$ (see [6],[12, 13]). Assume $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$. Let u be its harmonic extension, i.e. the complex valued function agreeing with g on $\partial\Omega$ such that $\Delta u = 0$ in Ω. Consider the 1-form $\omega = (iu, \partial_1 u)dx_1 + (iu, \partial_2 u)dx_2$ that we write in shorthand (iu, du). Then

$$\deg(g, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} \omega = \frac{1}{2\pi} \int_{\Omega} d\omega.$$

But $d\omega = (idu, du) = 2 \operatorname{jac} u$ where $\operatorname{jac} u$ is the jacobian determinant of u (seen as a map from Ω to $\mathbb{R}^2$) thus

$$\deg(g, \partial\Omega) = \frac{1}{\pi} \int_{\Omega} \operatorname{jac} u(x)\, dx. \tag{3.2}$$

Now the map $g \to u$ that maps g to its harmonic extension is continuous from $H^{\frac{1}{2}}$ to H^1 and the integral above is continuous in the H^1 norm of u. Therefore the degree defined on $C^\infty(\partial\Omega, \mathbb{S}^1)$ is continuous in the $H^{\frac{1}{2}}$ norm. This allows to continuously extend the degree on the closure of $C^\infty(\partial\Omega, \mathbb{S}^1)$ in the $H^{\frac{1}{2}}$-norm. A non trivial but true fact (see [42]) is that this closure is $H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^1)$, the space of maps $g \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{C})$ such that $|g| = 1$ a.e.

3.1.2 Properties

The properties of the degree for maps in $H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^1)$ are similar to those of the degree for smooth maps. We refer to [12, 13] for some of the proofs.

Property 1 The degree is an integer, this is trivial.
Property 2 $\deg(g, \partial\Omega)$ can be computed by (3.2) for any extension $u \in H^1(\Omega, \mathbb{C})$ of g.
Property 3 For $g \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^1)$ there exists $u \in H^1(\Omega, \mathbb{S}^1)$ agreeing with g on $\partial\Omega$ if and only if $\deg(d, \partial\Omega) = 0$.

3.2 The co-area formula

The co-area formula is a generalization of Fubini's Theorem. We will state it in a simple form (see [22] for details).

Assume f is a function of two variables defined in a neighbourhood of x and such that $\nabla f(x) \neq 0$ and let $t_0 = f(x)$. Then the implicit function Theorem implies the existence of a neighbourhood U of x, of two open intervals I, J such that $t_0 \in I$ and of a diffeomorphism $\varphi : I \times J \to U$ such that $f(\varphi(t,s)) = t$. If g is any integrable function on U

$$\int_U g(x)\,dx = \int_{t\in I}\int_{s\in J} g(\varphi(t,s))|\operatorname{jac}\varphi(t,s)|\,ds\,dt. \tag{3.3}$$

Differentiating $f(\varphi(t,s)) = t$ w.r.t. t and s we find

$$\nabla f(\varphi(t,s)).\partial_t\varphi(t,s) = 1, \quad \nabla f(\varphi(t,s)).\partial_s\varphi(t,s) = 0.$$

Using a little linear algebra this implies

$$|\operatorname{jac}\varphi(t,s)| = \frac{|\partial_s\varphi(t,s)|}{|\nabla f(\varphi(t,s))|}.$$

Replacing in (3.3) yields

$$\int_U g(x)\,dx = \int_{t\in I}\int_{s\in J} g(\varphi(t,s))\frac{|\partial_s\varphi(t,s)|}{|\nabla f(\varphi(t,s))|}\,ds\,dt.$$

The inner integral is the integral of the function $g(x)/|\nabla f(x)|$ over the curve $\gamma_t = \{x \in U \mid f(x) = t\}$ — parametrized by $s \to \varphi(t,s)$ — w.r.t. the measure $|\partial_s\varphi|\,ds$, which is precisely arclength. We write this as

$$\int_U g(x)\,dx = \int_{t\in\mathbb{R}}\int_{\{x\in U|f=t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^1\,dt, \tag{3.4}$$

where $\mathscr{H}^1$ denotes the one dimensional Hausdorff measure or arclength and of course if $\{f = t\} = \varnothing$ the corresponding integral is taken to be zero.

The above argument can be used to prove the following

Theorem 1. *Let Ω be an open set in $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$ a C^1 function with nonvanishing gradient. Then for any $g \in L^1(\Omega)$*

$$\int_\Omega g(x)\,dx = \int_{t\in\mathbb{R}}\int_{\{f=t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^1\,dt.$$

The above result easily generalizes to dimensions $n \neq 2$ simply replacing $\mathscr{H}^1$ by $\mathscr{H}^{n-1}$. The result also generalizes to $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$ where $k \leq n$. In this case — letting df and df^* denote the differential of f and its transpose — $|\nabla f|$ must be replaced by $\mathrm{jac}_k(f) = \sqrt{\det(df \circ df^*)}$, and $\mathscr{H}^1$ by $\mathscr{H}^{n-k}$ (see [22]). The assumption that f is C^1 may also be relaxed to f Lipschitz.

3.3 Sard's Theorem

Sard's Theorem is a very important tool in analysis: it states that most level sets of a smooth enough map $f : \mathbb{R}^n \to \mathbb{R}^k$ are as expected submanifolds of $\mathbb{R}^n$ of dimension $n - k$.

Definition 4. *Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^k$ a C^1 function. We say $t \in \mathbb{R}^k$ is a regular value of f if $\forall x \in \Omega$ we have*

$$f(x) = t \implies \mathit{rank}(df_x) = k.$$

in particular if the level-set $\{x \in \Omega \mid f(x) = t\}$ is empty then t is a regular value of f. If a value is not regular, we say it is critical.

From the implicit function Theorem, a sufficient condition for the level-set $\{x \in \Omega \mid f(x) = t\}$ to be a $n-k$-dimensional C^1 submanifold of Ω is that t be a regular value of f. This is the object of Sard's Theorem.

Theorem 2. *Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^k$ a C^p map, where $p \geq \max(1, n-k+1)$. Then almost every $t \in \mathbb{R}$ is a regular value of f.*

Remark 3. *In the case $n = 2$, $k = 1$, Sard's Theorem requires that the function f be C^2. This condition is sharp in the sense that there exists a C^1 function defined on $\mathbb{R}^2$ for which any $0 \leq t \leq 1$ is a critical value. (see [17])*

We refer to [17] for the proof of this theorem.

Chapter 4

The ball construction method

As we will see this method allows to associate "vortex-balls" to a map $u : \Omega \to \mathbb{C}$ such that $F_\varepsilon(u)$ satisfies certain bounds. A version works also for the Ginzburg-Landau model with magnetic field. Vortex-balls in this sense must obey three properties: 1) they must contain the topological zeroes of u, 2) their radius must be small when ε is small, 3) they must contain the right amount of energy.

4.1 $\mathbb{S}^1$-valued maps on perforated domains

We first adress a model problem. Assume ω is a compact subset of a bounded smooth domain $\Omega \subset \mathbb{R}^2$ and $u : \Omega \setminus \omega \to \mathbb{S}^1$. Let $d = \deg(u, \partial\Omega)$. If ω is empty, that is if u is defined on all of Ω and if u is continuous then d must be zero (use (3.2) together with $|u| = 1$). It is less obvious but nevertheless true (see [5]) that if u is in $H^1(\Omega, \mathbb{S}^1)$ (i.e. $u \in H^1(\Omega, \mathbb{R}^2)$ and $|u| = 1$ a.e.) then d must be zero. An intuitive justification for this is that $W^{1,p}$ embeds into the space of continous functions for any $p > 2$, H^1 therefore appears as a borderline case. We thus expect that if d is non zero and ω is small, then the H^1 norm of u must be large. This is the result we prove in this section.

4.1.1 Radius of a set

We define here what will be our measure of size for a compact set $\omega \subset \mathbb{R}^2$.

Definition 5. *The* radius *of a compact set $\omega \subset \mathbb{R}^2$ is the infimum over all finite coverings of ω by open balls $B_1, \dots, B_k$ of the sum $r_1 + \dots + r_k$, where r_i is the radius of B_i. We write $r(\omega)$ for this quantity.*

Remark 4. *Note that in this definition we may assume the covering is by balls with disjoint closures. Indeed if $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$ satisfy $\overline{B}_1 \cap \overline{B}_2 \neq \varnothing$ then*

$$B_1 \cup B_2 \subset B\left(\frac{r_1 a_1 + r_2 a_2}{r_1 + r_2}, r_1 + r_2\right). \tag{4.1}$$

Using this to group together intersecting balls, a finite covering may be replaced by a covering by balls with disjoint closures leaving the sum of radii unchanged. We leave this as well as (4.1) to check as an exercise.

There is a relationship between radius and perimeter. Recall the definition of 1-dimensional Hausdorff measure.

Definition 6. *The 1-dimensional Hausdorff measure of $A \subset \mathbb{R}^2$ is*

$$\mathscr{H}^1(A) = \liminf_{\varepsilon \to 0} \mathscr{H}^1_\varepsilon(A)$$

where $\mathscr{H}^1_\varepsilon(A)$ *is the infimum of the quantity* $\sum_{i\in\mathbb{N}} 2r_i$ *over all coverings* $\{B_i\}_{i\in\mathbb{N}}$ *of* A *by open balls of radii* $\{r_i\}_{i\in\mathbb{N}}$ *such that* $\sup_i r_i \le \varepsilon$ *.*

Proposition 9. *Assume* ω *is a compact subset of* $\mathbb{R}^2$*. Then* $2r(\omega) \le \mathscr{H}^1(\partial\omega)$*.*

Proof. It suffices to show that if $\{B_i\}_{i\in\mathbb{N}}$ is any covering of $\partial\omega$ by open balls of radii $\{r_i\}_{i\in\mathbb{N}}$, then $r(\omega) \le \sum_i r_i$. Since $\partial\omega$ is compact we may extract a finite covering, and then using remark 4 we assume the balls have disjoint interiors. In particular $A = \mathbb{R}^2 \setminus \cup_{i=1}^k B_i$ is connected. Now if $B_1, \dots, B_k$ cover $\partial\omega$, we claim they cover ω and therefore $r(\omega) \le \sum_i r_i$, which concludes the proof. The claim follows by noting that A — which is connected — intersects the complement of ω because ω is bounded. Thus if A intersected ω it would also intersect $\partial\omega$, which is impossible from the definition of A. Thus $\omega \subset \mathbb{R}^2 \setminus A = \cup_{i=1}^k B_i$. □

4.1.2 Main Theorem

In the following, we will write

$$E(\Omega, u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx$$

for the energy of a map u defined on Ω.

Theorem 3. *Let* ω *be a compact subset of a bounded domain* $\Omega \subset \mathbb{R}^2$*. Then for any* $\eta > \alpha > r(\omega)$ *there exist a collection* $\{B_i\}_{1\le i\le k}$ *of disjoint open balls with radii* $\{r_i\}_{1\le i\le k}$ *such that 1)* $r_1 + \cdots + r_k \le \eta$*, 2)* $\omega \subset \cup_i B_i$*, 3) for any* $u : \Omega\setminus\omega \to S^1$ *and any* $1 \le i \le k$

$$E(B_i \setminus \omega, u) \ge \pi |d_i| \log\frac{\eta}{\alpha}, \tag{4.2}$$

where $d_i = \deg(u, \partial B_i)$ *if* $B_i \subset \Omega$ *and* $d_i = 0$ *otherwise.*

Remark 5. *In this result* α *is present to insure that a* finite *collection of balls* B_i *exists. It could probably be shown by taking for* ω *some Cantor set that* α *cannot simply be replaced by* $r(\omega)$*.*

The intuitive reason as well as the basic element for the proof of this theorem is given by

Lemma 6. *Let* $C_{r,R} = \{x \in \mathbb{R}^2 \mid r < |x| < R\}$*. Then for any* $0 < r < R$ *and any* $u : C_{r,R} \to \mathbb{S}^1$

$$E(u, C_{r,R}) \ge \pi |d|^2 \log\frac{R}{r},$$

where $d = \deg(u, \gamma_s)$ *where* $\gamma_s = \{x \in \mathbb{R}^2 \mid |x| = s\}$ *is any circle such that* $r < s < R$*.*

Proof. Let (ρ, θ) be polar coordinates on the annulus $C_{r,R}$, then

$$|\nabla u|^2 = |\partial_\rho u|^2 + \rho^{-2}|\partial_\theta u|^2.$$

Therefore

$$E(u, C_{r,R}) \ge \frac{1}{2}\int_r^R \int_0^{2\pi} \frac{1}{s^2}|\partial_\theta u|^2 s\, d\theta\, ds.$$

But

$$\int_0^{2\pi} |\partial_\theta u| d\theta \ge \left|\int_0^{2\pi} (iu, u_\theta) d\theta\right| = 2\pi |d|,$$

thus, using the Cauchy-Schwarz inequality

$$E(u, C_{r,R}) \ge \frac{1}{2}\int_r^R \frac{2\pi |d|^2}{s} ds.$$

Integrating yields the result for sufficiently smooth u, for general $u \in H^1$ the result follows by a density argument. □

The proof of the Theorem involves summing lower bounds on well chosen disjoint annuli.

Proof. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and ω a compact subset of Ω and $\alpha > r(\omega)$. From the definition of the radius of ω there exists a covering of ω by a finite collection $\mathscr{B}(0)$ of disjoint open balls whose sum of radii is less than α. We wish to construct for each $t \geq 0$ a collection $\mathscr{B}(t)$ which satisfies properties 1), 2), 3) of the Theorem with $\eta = \exp(t)\alpha$ — we will then say $\mathscr{B}(t)$ is appropriate.

The collection $\mathscr{B}(0)$ clearly is appropriate. Assume $\mathscr{B}(t_0)$ is appropriate for some $t_0 \geq 0$. Then two cases may occur.

Case 1, Merging

If two balls in $\mathscr{B}(t_0)$ have intersecting closures — call them B_1, B_2 and r_1, r_2 their radii — we may group them in a larger ball B with radius $r = r_1 + r_2$ using remark 4. We then remove B_1, B_2 from $\mathscr{B}(t_0)$ and add B. Repeating this operation enough times we get a family $\mathscr{B}'(t_0)$ of balls with nonintersecting closures. Is the collection $\mathscr{B}'(t_0)$ appropriate? It certainly still satisfies properties 1) and 2) of the Theorem. For 3), take $u : \Omega \setminus \omega \to \mathbb{S}^1$ and $B \in \mathscr{B}'(t_0)$. If $B \not\subset \Omega$ then (4.2) is trivially true. If B was originally in $\mathscr{B}(t_0)$ then (4.2) is also trivial. In the remaining case there are $B_1, \ldots, B_k \in \mathscr{B}(t_0)$ such that $\cup_i B_i \subset B$ and $B \cap \omega \subset \cup_i B_i$. Then $\deg(u, \partial B) = \sum_i \deg(u, \partial B_i)$ hence (4.2) is verified. We are in position to apply the following.

Case 2, Growing

If all the balls in $\mathscr{B}(t_0)$ have disjoint closures, then they will remain disjoint if their radius is increased by a small amount. In other words there exists $\lambda > 1$ such that the balls $\{\lambda B\}_{B \in \mathscr{B}(t_0)}$ are disjoint, where λB denotes the ball with *same center* as B and radius multiplied by λ. Let $\mathscr{B}(t + \delta) = \{\lambda B\}_{B \in \mathscr{B}(t_0)}$, where $\lambda = \exp(\delta)$. This collection clearly satisfies properties 1) and 2). Now take $u : \Omega \setminus \omega \to \mathbb{S}^1$ and $\lambda B \in \mathscr{B}(t + \delta)$. If $\lambda B \not\subset \Omega$ then (4.2) is true. If not $E(u, \lambda B \setminus \omega) = E(u, \lambda B \setminus B) + E(u, B \setminus \omega)$ and Lemma 6 yields $E(u, \lambda B \setminus B) \geq \pi |d|^2 \log \lambda$, where $d = \deg(u, \partial B)$. Since $E(u, B \setminus \omega) \geq \pi |d| t_0$ and $d^2 \geq |d|$ It follows that

$$E(u, \lambda B \setminus \omega) \geq \pi |d| (t_0 + \delta)$$

and therefore $\mathscr{B}(t_0 + \delta)$ is appropriate.

It is easy to see that by growing and merging, an appropriate collection of balls may be constructed for all $t \in \mathbb{R}_+$ and thereby prove the Theorem. It is important to note here that the number of balls in $\mathscr{B}(0)$ is finite, that the number of balls decreases after merging and remains constant through growing. □

4.1.3 Variants

Theorem 3 may be generalized in several ways: in higher dimensions or on a riemannian manifold for instance, and we let the reader guess what the result should be in these cases. We state below without proof a gauge invariant version of it that will be useful to us later on. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and let ω be a compact subset of Ω. For $u : \Omega \setminus \omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$ we define

$$E(u, A, \Omega \setminus \omega) = \frac{1}{2} \iint_{\Omega \setminus \omega} |\nabla_A u(x)|^2 \, dx, \qquad F(A, \Omega) = \frac{1}{2} \iint_{\Omega} h^2(x) \, dx,$$

where $\nabla_A u = \nabla u - iAu$ is the covariant gradient of u w.r.t. the connection A and $h = \partial_1 A_2 - \partial_2 A_1$.

The equivalent of Lemma 6 in this case is

Lemma 7. *Let $C_{r,R} = \{x \in \mathbb{R}^2 \mid r < |x| < R\}$ and $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Then for any $0 < r < R$ and any $u : C_{r,R} \to \mathbb{S}^1$, $A : B_R \to \mathbb{R}^2$*

$$E(u, A, C_{r,R}) + (R - r) F(A, B_R) \geq \pi |d|^2 \left(\log \frac{R}{r} - \frac{1}{2}(R - r) \right),$$

where $d = \deg(u, \gamma_s)$ *where* $\gamma_s = \{x \in \mathbb{R}^2 \mid |x| = s\}$ *is any circle such that* $r < s < R$.

The proof can be found in [32].

Remark 6. *Note that the curvature term has to be integrated over the whole ball.*

The lower bound above is clearly not interesting if R *is large since the right-hand side becomes negative. We will therefore use it for* $R < 1$ *only, which insure that the right-hand side is positive.*

Following the same line of proof as in Theorem 3 we obtain

Theorem 4. *Let* ω *be a compact subset of a bounded domain* $\Omega \subset \mathbb{R}^2$. *Then for any* $1 > \eta > \alpha > r(\omega)$ *there exist a collection* $\{B_i\}_{1 \le i \le k}$ *of disjoint open balls with radii* $\{r_i\}_{1 \le i \le k}$ *such that 1)* $r_1 + \cdots + r_k \le \eta$, *2)* $\omega \subset \cup_i B_i$, *3) for any* $u : \Omega \setminus \omega \to S^1$, $A : \Omega \to \mathbb{R}^2$ *and any* $1 \le i \le k$

$$E(u, A, B_i \setminus \omega, u) + r_i F(A, B_i) \ge \pi |d_i| \left(\log \frac{\eta}{\alpha} - \frac{r_i}{2} \right), \tag{4.3}$$

where $d_i = \deg(u, \partial B_i)$ *if* $B_i \subset \Omega$ *and* $d_i = 0$ *otherwise.*

4.2 Vortex Balls

In this section we show that appropriate a-priori bounds on the Ginzburg-Landau energy with or without magnetic field imply the existence of a vortex structure.

4.2.1 Using the coarea formula

Here Ω is a smooth bounded domain in $\mathbb{R}^2$. We begin with an elementary result

Lemma 8. *Let* u *be complex-valued function differentiable at a point* $x \in \mathbb{R}^2$ *such that* $u(x) \neq 0$. *Then*

1) $|\nabla u|(x) \ge |\nabla |u||(x)$.

2) $|\nabla u - iAu(x)|(x) \ge |\nabla |u||(x)$, *for any* A.

We deduce the following

Corollary 1. *For any* $u \in C^1(\Omega, \mathbb{C})$, *any* $A \in C^0(\Omega, \mathbb{C})$ *and any* $\varepsilon > 0$

$$\frac{1}{2} \int_\Omega |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1 - \rho^2)^2 \le \min \left(F_\varepsilon(u, \Omega), J_\varepsilon(u, A, \Omega) \right), \tag{4.4}$$

where $\rho = |u|$.

Proof. The integral on the left is well defined: since $x \to |x|$ is lipschitz and u is C^1 then ρ is Lipschitz and therefore differentiable a.e. If $\rho(x) \neq 0$ it is clear from the preceding Lemma that the integrand on the left hand side of (4.4) is smaller then the integrand of the Ginzburg-Landau energy with or without magnetic field. Besides it is known (even for Sobolev functions) that $\nabla \rho = 0$ a.e. on the set where $\rho = 0$. This allows to conclude. □

Using the coarea formula, we are the able to prove

Proposition 10. *There exists* $\alpha_0(\Omega) > 0$ *such that for any* $M, \varepsilon, \delta > 0$ *satisfying* $\varepsilon M / \delta^2 < \alpha_0$ *and* $\varepsilon < 1$, *any* $u \in C^2(\overline{\Omega}, \mathbb{C})$ *satisfying either* $F_\varepsilon(u, \Omega) \le M$ *or* $J_\varepsilon(u, A, \Omega) \le M$ *for some* $A \in C^0(\Omega, \mathbb{C})$

$$r\left(\{x \in \overline{\Omega} \mid |u(x)| \le 1 - \delta\} \right) \le C \frac{\varepsilon M}{\delta^2},$$

for some universal constant C.

Remark 7. *The requirement that* u *be* C^2 *is needed to apply Sard's Theorem at some point in the proof.*

Proof. Let $\rho = |u|$. Using the Corollary above, we have

$$\frac{1}{2}\int_{\Omega} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \le M.$$

For practical reasons, we need ρ to be defined on a larger domain Ω' such that $\overline{\Omega} \subset \Omega'$. It is certainly possible to do this in a way such that

$$\frac{1}{2}\int_{\Omega'} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \le 2M.$$

which by Cauchy-Schwarz yields

$$\frac{\sqrt{2}}{2}\int_{\Omega'} |\nabla\rho|\frac{(1-\rho^2)}{\varepsilon} \le 2M.$$

and then using the coarea formula (Theorem 1), and letting $\Omega'' = \{x \in \Omega' \mid \nabla\rho(x) \neq 0\}$,

$$\frac{\sqrt{2}}{2}\int_{t\in\mathbb{R}} \frac{(1-t^2)}{\varepsilon}\mathscr{H}^1\left(\{x \in \Omega'' \mid \rho(x) = t\}\right)\, dt \le 2M. \tag{4.5}$$

From Sard's Theorem (Theorem 2) and since ρ is C^2 where $\rho \neq 0$, almost every $t \in \mathbb{R}$ is a regular value of ρ i.e. $\{\rho = t\} \subset \Omega''$. Then — since $(1-t^2) \ge \delta/2$ for any $t \in (1-\delta, 1-\delta/2)$ — the mean-value theorem and (4.5) imply the existence of some $t \in (1-\delta, 1-\delta/2)$ which is a regular vaue of ρ and such that

$$\mathscr{H}^1\left(\{x \in \Omega' \mid \rho(x) = t\}\right)\, dt \le 2M\frac{2}{\sqrt{2}} \times \frac{2\varepsilon}{\delta} \times \frac{2}{\delta}. \tag{4.6}$$

on the other hand, since

$$\frac{1}{4\varepsilon^2}\int_{\Omega'} (1-\rho^2)^2 \le M$$

the real number t also satisfies

$$|\{x \in \Omega' \mid \rho(x) \le t\}| \le \frac{4M\varepsilon^2}{\delta^2} \le \frac{4M\varepsilon}{\delta^2}, \tag{4.7}$$

since from now on we assume that $\varepsilon < 1$. We also assume $\varepsilon M/\delta^2 < \alpha_0$, for some $\alpha_0 > 0$ to be specified later. Let $\omega = \{x \in \overline{\Omega} \mid \rho(x) \le t\}$. Equation (4.6) is not sufficient to bound $r(\omega)$ — which is our aim — but it will be with the help of (4.7). Equation (4.6) implies the existence of a finite collection of disjoint balls $B_1, \dots, B_k$ with radii $r_1, \dots, r_k$ such that

$$\{x \in \overline{\Omega} \mid \rho(x) = t\} \subset \cup_{i=1}^k B_i, \quad \sum_{i=1}^k r_i \le C\frac{M\varepsilon}{\delta^2}, \tag{4.8}$$

where C is a universal constant. Since Ω is a smooth domain, there exists $r_0(\Omega) > 0$ such that any finite collection of balls $\{B(a_i, r_i)\}$ satisfying $\sum_i r_i \le r_0$ is such that $\Omega \setminus \cup_i B(a_i, r_i)$ is connected (one may choose as r_0 any lower bound for the radius of a ball tangent to $\partial\Omega$ at two distinct points). Using (4.8) this implies that there exists $\alpha_0 > 0$ depending on Ω only such that $\tilde{\Omega} = \Omega \setminus \cup_{i=1}^k B_i$ is connected. Then either $\rho > t$ on $\tilde{\Omega}$ or $\rho < t$ on $\tilde{\Omega}$. If α_0 is small enough, the latter is impossible: the bound (4.8) insures that $|\tilde{\Omega}| \ge |\Omega|/2$ if α_0 is small enough (depending on Ω) while (4.7) implies that $|\{\rho \le t\}| < |\Omega|/2$, again for small enough α_0.

It follows that there exists $\alpha_0(\Omega) > 0$ such that $\{\rho \le t\} \subset \cup_i B_i$ and therefore

$$r(\{\rho \le t\}) \le C\frac{\varepsilon M}{\delta^2}.$$

Since $t \ge 1 - \delta$ the Proposition is proved. □

4.2.2 Main result

We are now able to prove

Theorem 5. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $M > 0$. There exists $\varepsilon_0(\Omega, M) > 0$ and $C(\Omega, M) > 0$ such that for any $\varepsilon < \varepsilon_0$, any $R < |\log\varepsilon|^{-2}$ and any $u \in C^2(\overline{\Omega}, \mathbb{C})$, $A \in C^1(\Omega, \mathbb{R}^2)$ satisfying $J_\varepsilon(u, A, \Omega) \le M|\log\varepsilon|^2$ with $h_{ext} \le M|\log\varepsilon|$ there exists a family of balls $\{B(a_i, r_i)\}_{1\le i\le k}$ satisfying the following properties.*

1. $\{x \in \Omega \mid |u| \le 1 - |\log\varepsilon|^{-2}\} \subset \cup_{i=1}^k B(a_i, r_i)$.
2. $\sum_{i=1}^k r_i \le R$.
3. *Writing* $u = \rho e^{i\varphi}$ *and* $h = \operatorname{curl} A$

$$\frac{1}{2}\int_{B(a_i,r_i)} \rho^2|\nabla\varphi - A|^2 + |h - h_{ext}|^2 \ge \pi|d_i|(\log\frac{R}{\varepsilon} - C\log|\log\varepsilon|), \tag{4.9}$$

where d_i is the degree of the map $u/|u|$ restricted to $\partial B(a_i, r_i)$ if $B(a_i, r_i) \subset \Omega$ and $d_i = 0$ otherwise.

Proof. Throughout this proof C denotes a generic positive constant depending eventually on Ω, M and N. First we use Proposition 10, with M replaced by $M|\log\varepsilon|$ and $\delta = ||\log\varepsilon||^{-2}$ It follows that there exist $\varepsilon_0, C > 0$ such that for any $\varepsilon < \varepsilon_0$ and any (u, A) satisfying $J_\varepsilon(u, A, \Omega) \le M|\log\varepsilon|^2$, the set $\omega = \{x \in \overline{\Omega} \mid |u(x)| \le 1 - ||\log\varepsilon||^2\}$ is such that

$$r(\omega) \le C_0 M\varepsilon|\log\varepsilon|^4, \tag{4.10}$$

where C_0 is a universal constant.

Next we use Theorem 4 with $\alpha = 2C_0M\varepsilon|\log\varepsilon|^4$ and $\eta = R$ to the map $v = u/|u|$ and the connection A. It follows that there exists a family of balls $\{B(a_i, r_i)\}_{1\le i\le k}$ satisfying properties 1 and 2 above and on each ball the lower bound

$$\frac{1}{2}\int_{B(a_i,r_i)\setminus\omega} |\nabla\varphi - A|^2 + \frac{r_i}{2}\int_{B(a_i,r_i)} h^2 \ge \pi|d_i|\left(\log\frac{R}{2C_0M|\log\varepsilon||\log\varepsilon|^4\varepsilon} - \frac{r_i}{2}\right),$$

which we may write as

$$\frac{1}{2}\int_{B(a_i,r_i)\setminus\omega} |\nabla\varphi - A|^2 + \frac{r_i}{2}\int_{B(a_i,r_i)} h^2 \ge \pi|d_i|\left(\log\frac{R}{\varepsilon} - C\log|\log\varepsilon|\right), \tag{4.11}$$

if ε is small enough.

Since $\rho \ge 1 - |\log\varepsilon|^{-2}$ on $B(a_i, r_i) \setminus \omega$, we have for ε small enough

$$\frac{\rho^2 - 1}{\rho^2} \ge -C|\log\varepsilon|^{-2}$$

on this set. Therefore

$$\int_{B(a_i,r_i)\setminus\omega} (\rho^2 - 1)|\nabla\varphi - A|^2 \ge -C|\log\varepsilon|^{-2}\int_{B(a_i,r_i)\setminus\omega} \rho^2|\nabla\varphi - A|^2 \ge -CJ_\varepsilon(u, A) \ge -CM. \tag{4.12}$$

Adding to (4.11) yields, changing C if necessary

$$\frac{1}{2}\int_{B(a_i,r_i)\setminus\omega} \rho^2|\nabla\varphi - A|^2 + \frac{r_i}{2}\int_{B(a_i,r_i)} h^2 \ge \pi|d_i|\left(\log\frac{R}{\varepsilon} - C\log|\log\varepsilon|\right). \tag{4.13}$$

Since $r_i \le R < |\log\varepsilon|^{-2}$ we may replace if ε is small enough the coefficient $r_i/2$ in front of the integral of h^2 by $1/2$. We also note that by Cauchy-Schwarz

$$-\int_{B(a_i,r_i)} h_{\text{ext}} h \ge -C h_{\text{ext}} r_i \left(\int_{B(a_i,r_i)} h^2 \right)^{\frac{1}{2}} \ge -C.$$

Adding to (4.13)

$$\frac{1}{2}\int_{B(a_i,r_i)\setminus\omega} \rho^2 |\nabla\varphi - A|^2 + \frac{1}{2}\int_{B(a_i,r_i)} (h - h_{\text{ext}})^2 \ge \pi |d_i| \left(\log\frac{R}{\varepsilon} - C\log|\log\varepsilon| \right).$$

The theorem is proved. □

Definition 7. *We will call any family of balls satisfying properties 1, 2, 3 of Theorem5 a family of* vortex balls *associated to* (u, A). *The measure*

$$2\pi \sum_{i=1}^{k} d_i \delta_{a_i}$$

will be called the corresponding vorticity measure.

Chapter 5

The Jacobian estimate.

We have shown in the preceding chapter that the a-priori bound $J_\varepsilon(u, A, \Omega) \leq M|\log\varepsilon|^2$ was enough to associate to (u, A) a family of vortex-balls, satisfying some properties that could seem rather arbitrary. We also described a possible construction of these vortex balls. Our aim here is to show that the construction is not arbitrary after all. More precisely the vorticity measures associated to a configuration (u, A) such that $J_e p(u, A, \Omega) \leq M|\log\varepsilon|^2$ by Theorem 5 are close to the jacobian determinant $\operatorname{jac} u$ when ε is small and in a weak enough norm. The first proof of this may be found in [33], although a bit hidden. The result was then improved and more clearly stated in [25]. The jacobian determinant is *not* a gauge-invariant quantity, therefore it is hopeless to prove any estimate concerning it except if we choose a particuliar gauge. We prefer instead to replace $\operatorname{jac} u$ by t! he quantity $\operatorname{curl}(iu, \nabla_A u) + h$ which is in a sense the gauge-invariant equivalent of the jacobian.

5.1 Main theorem

Our aim is to prove

Theorem 6. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, let $M > 0$, $0 < \gamma \leq 1$. Then for any $0 \leq \delta < 1/2$, any $u \in C^1(\overline{\Omega}, \mathbb{C})$, $A \in C^0(\overline{\Omega}, \mathbb{R}^2)$ and any collection of disjoint balls $\{B(a_i, r_i)\}_{1 \leq i \leq k}$ satisfying the following properties*

1. *$J_\varepsilon(u, A, \Omega) \leq M$,*

2. *$|u| \leq 1$ in Ω and $\{|u| \leq 1 - \delta\} \subset \cup_i B(a_i, r_i)$,*

there is a decomposition

$$2\pi \sum_{i=1}^{k} d_i \delta_{a_i} - \operatorname{curl}(iu, \nabla_A u) - h = \alpha + \beta,$$

where α and β satisfy

$$\|\alpha\|_{(H_0^1)'} \leq CM\delta^2, \quad \|\beta\|_{(C_0^\gamma)'} \leq CMR^\gamma. \tag{5.1}$$

Here C is a universal constant, $h = \operatorname{curl} A$, $R = \sum_i r_i$, C_0^γ denotes the space of h?der continuous functions of exponent γ that vanish on $\partial\Omega$ and $d_i = \deg(u, \partial B(a_i, r_i))$ if $B(a_i, r_i) \subset \Omega$ while $d_i = 0$ otherwise.

The proof is in two steps. First by a very elementary argument we may assume $|u| = 1$ outside of $\cup_i B(a_i, r_i)$, a hypothesis under which the proof is then particularly easy.

5.2 The case $|u| = 1$ on $\Omega \setminus \cup_i B(a_i, r_i)$

In this case $\mathrm{curl}(iu, \nabla_A u) + h \equiv 0$ in $\Omega \setminus \cup_i B(a_i, r_i)$ and therefore for any $\xi \in C_0^\gamma$,

$$\int_\Omega \xi \,(\mathrm{curl}(iu, \nabla_A u) + h) = \sum_i \int_{B(a_i,r_i)} \xi \,(\mathrm{curl}(iu, \nabla_A u) + h)\,. \tag{5.2}$$

First we assume $B(a_i, r_i) \cap \partial\Omega = \varnothing$. For all $x \in B(a_i, r_i)$ we let $f(x) = \xi(x) - \xi(a_i)$. Then

$$|f(x)| \le r_i^\gamma \|\xi\|_{C^\gamma}. \tag{5.3}$$

A straightforward computation shows that $\mathrm{curl}(iu, \nabla_A u) = 2\nabla|u| \wedge \nabla_A u - |u|^2 h$ thus the L^1 norm of $\mathrm{curl}(iu, \nabla_A u) + h$ on $B(a_i, ri)$ is less then $2J_\varepsilon(u, A, B(a_i, r_i))$. Together with (5.3), this yields

$$\int_{B(a_i,r_i)} f\,(\mathrm{curl}(iu, \nabla_A u) - h) \le 2r_i^\gamma \|\xi\|_{C^\gamma} J_\varepsilon(u, A, B(a_i, r_i)). \tag{5.4}$$

On the other hand

$$\int_{B(a_i,r_i)} \mathrm{curl}(iu, \nabla_A u) + h = \int_{\partial B(a_i,r_i)} (iu, \partial_\tau u - iA.\tau u) + A.\tau.$$

Since $|u| = 1$ on $\partial B(a_i, r_i)$ we have $(iu, \partial_\tau u - iA.\tau u) + A.\tau = (iu, \partial_\tau u)$, the integral of which is precisely $2\pi d_i$. Therefore

$$\int_{B(a_i,r_i)} \xi(a_i)\,(\mathrm{curl}(iu, \nabla_A u) + h) = 2\pi d_i \xi(a_i).$$

Adding to (5.4) yields

$$\left| \int_{B(a_i,r_i)} \xi\,(\mathrm{curl}(iu, \nabla_A u) + h) - 2\pi d_i \xi(a_i) \right| \le 2r_i^\gamma \|\xi\|_{C^\gamma} J_\varepsilon(u, A, B(a_i, r_i)). \tag{5.5}$$

If $B(a_i, r_i)$ intersects $\partial\Omega$ then ξ vanishes on $B(a_i, r_i) \cap \partial\Omega$ and therefore $|\xi| \le 2r_i \|\xi\|_{C^\gamma}$ on $B(a_i, r_i) \cap \Omega$. Then as in the proof of (5.4) we find

$$\left| \int_{B(a_i,r_i)} \xi\,(\mathrm{curl}(iu, \nabla_A u) + h) \right| \le 4r_i^\gamma \|\xi\|_{C^\gamma} J_\varepsilon(u, A, B(a_i, r_i)). \tag{5.6}$$

Summing (5.5) or (5.6) over i according to wether $B(a_i, r_i)$ intersects $\partial\Omega$ or not, and taking the supremum over all $\xi \in C_0^\gamma$ we find

$$\left\| 2\pi \sum_{i=1}^k d_i \delta_{a_i} - \mathrm{curl}(iu, \nabla_A u) - h \right\|_{(C_0^\gamma)'} \le 2MR^\gamma. \tag{5.7}$$

In this case the term α in the Theorem is absent.

5.3 Reduction to the previous case

We now only assume $\{|u| \le 1 - \delta\} \subset \cup_i B(a_i, r_i)$, for some $0 \le \delta < 1/2$. We may reduce the proof to the latter case by defining a map v with modulus equal to 1 outside $\cup_i B(a_i, r_i)$ This is done by defining $\chi : [0, 1] \to [0, 1]$ by

$$\chi(x) = \begin{cases} \dfrac{x}{1-\delta} & \text{if } 0 \le x \le 1 - \delta \\ 1 & \text{if } x \ge 1 - \delta. \end{cases} \tag{5.8}$$

and letting

$$v := \chi(|u|)\frac{u}{|u|}, \tag{5.9}$$

We have

Proposition 11. *For any $0 \le \delta < 1/2$, any $u \in C^1(\overline{\Omega}, \mathbb{C})$ such that $|u| \le 1$ and any $A \in C^0(\overline{\Omega}, \mathbb{R}^2)$ we have*

$$\|\mathrm{curl}(iv, \nabla_A v) - \mathrm{curl}(iu, \nabla_A u)\|_{H_0^1(\Omega)'} \le C\delta^2 \int_\Omega |\nabla_A u|^2, \tag{5.10}$$

$$|\nabla_A v|^2 \le (1 + C\delta)|\nabla_A u|^2, \tag{5.11}$$

where v is defined by (5.8),(5.9).

Proof. We need the following estimate: the function χ satisfies for every $0 \le x \le 1$

$$|x^2 - \chi(x)^2| \le 6\delta x^2, \quad \chi'(x)^2 - 1 \le 2\delta. \tag{5.12}$$

The proof is a straightforward computation, using the fact that $0 \le \delta \le 1/2$.

We estimate the difference $\mathrm{curl}(iv, \nabla_A v) - \mathrm{curl}(iu, \nabla_A u)$. Assuming $|u(x)| \neq 0$, we may write locally $u = \rho e^{i\varphi}$ where ρ and φ are C^1 and real-valued. Then $(iu, \nabla_A u) = \rho^2(\nabla\varphi - A)$ while $(iv, \nabla_A v) = \chi(\rho)^2(\nabla\varphi - A)$. From (5.12) we deduce

$$|(iu, \nabla_A u) - (iv, \nabla_A v)| \le 6\delta\rho^2|\nabla\varphi - A| \le 6\delta|\nabla_A u|$$

and then

$$\|\mathrm{curl}(iv, \nabla_A v) - \mathrm{curl}(iu, \nabla_A u)\|_{H_0^1(\Omega)'} \le \int_\Omega |(iu, \nabla_A u) - (iv, \nabla_A v)|^2 \le 36\delta^2 \int_\Omega |\nabla_A u|^2. \tag{5.13}$$

For the second assertion we write $|\nabla_A u|^2$ as $\rho^2|\nabla\varphi - A|^2 + |\nabla\rho|^2$, and similarly for $|\nabla_A v|^2$ with ρ replaced by $\chi(\rho)$. Thus

$$|\nabla_A v|^2 - |\nabla_A u|^2 = (\rho^2 - \chi(\rho^2))|\nabla\varphi - A|^2 + (\chi'(\rho)^2 - 1)|\nabla\rho|^2,$$

which using (5.12) yields $|\nabla_A v|^2 - |\nabla_A u|^2 \le 6\delta|\nabla_A u|^2$. The proposition is proved □

The proof of the Theorem is almost complete. We define v as above. Since $|u| \ge 1 - \delta$ implies $|v| = 1$, we have $|v| = 1$ outside $\cup_i B(a_i, r_i)$. Therefore we may apply (5.7) to v to find

$$\left\| 2\pi \sum_{i=1}^{k} d_i \delta_{a_i} - \mathrm{curl}(iv, \nabla_A v) - h \right\|_{(C_0^\gamma)'} \le 2J_\varepsilon(v, A)R^\gamma \le CMR^\gamma, \tag{5.14}$$

Using (5.11). We let $\beta = \sum_i d_i \delta_{a_i} - \mathrm{curl}(iv, \nabla_A v) - h$ and $\alpha = \mathrm{curl}(iv, \nabla_A v) - \mathrm{curl}(iu, \nabla_A u)$. Then (5.10) yields

$$\|\alpha\|_{(H_0^1)'} \le CM\delta^2,$$

which together with (5.14) proves the Theorem.

5.4 A particular case

The jacobian estimate will be especially useful to us when combined with the vortex ball construction of Theorem 5.

Theorem 7. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $M > 0$, $1 < p < 2$. There exists $\varepsilon_0, C, N > 0$ possibly depending on Ω, M, p such that for any $\varepsilon < \varepsilon_0$ and any $u \in C^2(\overline{\Omega}, \mathbb{C})$, $A \in C^1(\Omega, \mathbb{R}^2)$ satisfying $J_\varepsilon(u, A, \Omega) \le M|\log\varepsilon|^2$ with $h_{ext} \le M|\log\varepsilon|$ there exists a family of balls $\{B(a_i, r_i)\}_{1 \le i \le k}$ satisfying the following properties.*

1. *$\{x \in \Omega \mid |u| \leq 1 - |\log\varepsilon|^{-2}\} \subset \cup_{i=1}^k B(a_i, r_i)$.*
2. *$\sum_{i=1}^k r_i \leq |\log\varepsilon|^{-N}$.*
3. *Writing $u = \rho e^{i\varphi}$ and $h = \operatorname{curl} A$, and letting d_i be the degree of the map $u/|u|$ restricted to $\partial B(a_i, r_i)$ if $B(a_i, r_i) \subset \Omega$ and $d_i = 0$ otherwise*

$$\frac{1}{2} \int\limits_{B(a_i,r_i)} \rho^2 |\nabla\varphi - A|^2 + |h - h_{ext}|^2 \geq \pi |d_i| (|\log\varepsilon| - C \log|\log\varepsilon|), \tag{5.15}$$

4. *With the same notation*

$$\left\| 2\pi \sum_{i=1}^k d_i \delta_{a_i} - \operatorname{curl}(iu, \nabla_A u) - h \right\|_{W_0^{-1,p}} \leq C |\log\varepsilon|^{-2}. \tag{5.16}$$

Proof. For any $q > 2$ the Sobolev space $W^{1,q}$ continuously embeds into C^γ for some $0 < \gamma < 1$ and $W_0^{1,q}$ embeds into C_0^γ. Therefore, for any $1 \leq p < 2$ the space $W_0^{-1,p}$ which is the dual of $W_0^{1,q}$, for $q = p/(p-1)$, continuously embeds into $(C_0^\gamma)'$ for some $0 < \gamma < 1$. On the other hand H_0^1 embeds into $W_0^{1,q}$ and therefore $W_0^{-1,p}$ embeds into $(H_0^1)'$. Choose $R < |\log\varepsilon|^{-2}$ and apply Theorem 5. We get vortex balls $\{B(a_i, r_i)_{1\leq i\leq k}$ with which in turn we may apply Theorem 6 to get

$$2\pi \sum_{i=1}^k d_i \delta_{a_i} - \operatorname{curl}(iu, \nabla_A u) - h = \alpha + \beta,$$

where

$$\|\alpha\|_{(H_0^1)'} \leq CM |\log\varepsilon|^{-2}, \quad \|\beta\|_{(C_0^\gamma)'} \leq CM |\log\varepsilon|^2 R^\gamma,$$

for some universal constant C. By choosing $R = |\log\varepsilon|^{-4/\gamma}$ we get

$$\|\alpha + \beta\|_{W_0^{-1,p}} \leq C |\log\varepsilon|^{-2},$$

where $C = C(M, \Omega)$. The lower bound (5.15) and other properties are easily verified. □

Remark 8. *The exponent N above can be chosen arbitrarily large but the larger N, the larger C will be and hence the worse the lower bound* (5.15).

Chapter 6

Computing H_{c1}

6.1 Meissner solution

Let Ω be a bounded smooth simply connected domain in $\mathbb{R}^2$. Then , contrary to the case $\Omega = \mathbb{R}^2$, the configuration $(u = 1, A = 0)$ is not a solution to (1.7) because the boundary conditions are not satisfied. What plays the role of this superconducting solution in our case is what we call the Meissner solution, an improper term since it is not a solution to (1.7), but only as far as A is concerned.

Proposition 12. *For any $h_{ext} > 0$, the infimum*

$$\inf_{A \in H^1(\Omega, \mathbb{R}^2)} J_\varepsilon(1, A)$$

is achieved by a unique connection $A = h_{ext} A_0$, where A_0 does not depend on h_{ext}. Moreover $A_0 = \nabla^\perp h_0$, with $h_0 = \operatorname{curl} A_0$ and h_0 is the solution to

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ \qquad\quad h_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Proof. Given a smooth A, we consider the following minimization problem

$$\min_{\varphi \in H^1(\Omega, \mathbb{R})} J_\varepsilon(1, A + \nabla\varphi).$$

The minimum is achieved by a function φ which solves

$$\begin{cases} \operatorname{div}(\nabla\varphi + A) = 0 & \text{in } \Omega \\ \nu \cdot (\nabla\varphi + A) = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore $A' = A + \nabla\varphi$ satisfies the Coulomb gauge condition and $J_\varepsilon(1, A') \le J_\varepsilon(1, A)$.

Now consider a minimizing sequence $(1, A_n)$ for $J_\varepsilon(1, A)$. By the above remark we may assume A_n satisfies the Coulomb gauge condition for every n and therefore the sequence $\{A_n\}_n$ is bounded in H^1 using (3). Thus modulo a subsequence we have weak H^1 convergence of $\{A_n\}_n$ to A and A is a minimizer because $A \to J_\varepsilon(1, A)$ is convex and continuous for the H^1 norm thus weakly lower semicontinuous. Uniqueness of the minimizer follows from the convexity.

The minimizer A satisfies the equation and boundary condition in (1.7) that are deduced from the variations of A i.e. $h = h_{\text{ext}}$ on $\partial\Omega$ and $-\nabla^\perp h = (iu, \nabla_A u)$ in Ω, where $h = \operatorname{curl} A$. Since $u = 1$ in our case this last equation becomes $-\nabla^\perp h = -A$. Taking the curl we find $\Delta h = h$ and therefore h solves

$$\begin{cases} -\Delta h + h = 0 & \text{in } \Omega \\ \qquad\quad h = h_{\text{ext}} & \text{on } \partial\Omega. \end{cases}$$

The conclusions of the Proposition follow from this and the relation $\nabla^\perp h = A$. $\square$

It follows easily from the above that

Proposition 13.

$$J_\varepsilon(1, h_{\text{ext}}A_0) = \frac{h_{\text{ext}}^2}{2}\|h_0 - 1\|^2_{H^1(\Omega)}. \tag{6.2}$$

Proof. If $(u, A) = (1, h_{\text{ext}}A_0)$ we have $\nabla_A u = -ih_{\text{ext}}A_0 = -ih_{\text{ext}}\nabla^\perp h_0$. Therefore

$$J_\varepsilon(u, A) = \frac{1}{2}\int_\Omega h_{\text{ext}}^2|\nabla h_0|^2 + h_{\text{ext}}^2(h_0 - 1)^2.$$

□

We will use the notation

$$J_0 = \frac{1}{2}\|h_0 - 1\|^2_{H^1(\Omega)}. \tag{6.3}$$

6.2 Energy in terms of vortices

In this section we consider the family of functionals J_ε for every $\varepsilon > 0$, where the parameter $h_{\text{ext}} = h_{\text{ext}}(\varepsilon)$ satisfies

$$h_{\text{ext}}(\varepsilon) \le C|\log\varepsilon|. \tag{6.4}$$

Then we consider a family $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that for every $\varepsilon > 0$

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \le Ch_{\text{ext}}(\varepsilon)^2 \tag{6.5}$$

and

$$|u_\varepsilon| \le 1 \quad \text{in } \Omega. \tag{6.6}$$

In view of (6.5), (6.4) we may apply Proposition 7 to $(u_\varepsilon, A_\varepsilon)$ with, say, $N = 10$ and conclude that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ a family $\mathscr{B}_\varepsilon$ of balls having the right properties can be defined. From now on we will drop the subscript ε for simplicity and thus we have configurations (u, A) — depending on ε — and a family of balls $\{B(a_i, r_i)\}_{1\le i\le k}$ — depending on ε also. As usual we let $h = \operatorname{curl} A$ and $\{d_i\}_{1\le i\le k}$ be the degrees of the vortices. We will also use the following notation

$$\omega = \cup_{i=1}^k B(a_i, r_i), \quad \widetilde{\Omega} = \Omega \setminus \omega. \tag{6.7}$$

From Proposition 7 we have

$$\sum_{i=1}^k r_i \le |\log\varepsilon|^{-10} \tag{6.8}$$

$$\frac{1}{2}\int_\Omega |\nabla_A u|^2 + (h - h_{\text{ext}})^2 \ge \pi\left(\sum_{i=1}^k |d_i|\right)(|\log\varepsilon| - C\log|\log\varepsilon|). \tag{6.9}$$

Also, letting

$$j = (iu, \nabla_A u) \tag{6.10}$$

we have using (6.6) that $|j|^2 \le |u|^2|\nabla_A u|^2 \le |\nabla_A u|^2$. Therefore

$$\frac{1}{2}\int_{\widetilde{\Omega}} |\nabla_A u|^2 + (h - h_{\text{ext}})^2 \ge \frac{1}{2}\int_{\widetilde{\Omega}} |j|^2 + (h - h_{\text{ext}})^2. \tag{6.11}$$

We define

$$j_0 = -\nabla^\perp h_0, \quad j_1 = j - h_{\text{ext}}j_0, \quad h_1 = h - h_{\text{ext}}h_0. \tag{6.12}$$

We have

Lemma 9.

$$\frac{1}{2}\int_{\widetilde{\Omega}} |j|^2 + (h - h_{ext})^2 \geq h_{ext}^2 J_0 + h_{ext} \int_{\Omega} j_1 \cdot j_0 + h_1(h_0 - 1) + o(1), \tag{6.13}$$

where $o(1)$ *denotes a function of* ε *which goes to zero as* $\varepsilon \to 0$.

Proof. Replacing h by $h_1 + h_{\text{ext}} h_0$ and j by $h_{\text{ext}} j_0 + j_1$ almost yields the result using the definition of J_0. The only problem is that the integral on the left is taken over $\widetilde{\Omega}$, instead of Ω on the right. It suffices therefore to prove that

$$I_1 = \frac{h_{\text{ext}}^2}{2}\int_{\omega} |j_0|^2 + (h_0 - 1)^2, \quad I_2 = h_{\text{ext}} \int_{\omega} j_1 \cdot j_0 + h_1(h_0 - 1)$$

are both $o(1)$.

The function h_0 does not not depend on ε and is C^1 thus $I_1 \leq C|\omega| h_{\text{ext}}^2$. Using (6.4) and (6.8) we conclude that I_1 is $o(1)$. On the other hand, using again the fact that h_0 does not depend on ε and is C^1,

$$|I_2| \leq C h_{\text{ext}} \int_{\omega} |j_1| + |h_1|.$$

Since $h_1 = h - h_{\text{ext}} h_0$ we have $\|h_1\|_{L^2(\Omega)}^2 \leq C h_{\text{ext}}^2$. Similarly, since $j_1 = j - h_{\text{ext}} j_0$ and since $\|j\|_{L^2(\Omega)}^2 \leq \|\nabla_A u\|_{L^2(\Omega)}^2 \leq C h_{\text{ext}}^2$ we have $\|j_1\|_{L^2(\Omega)}^2 \leq C h_{\text{ext}}^2$. Therefore by Cauchy-Schwartz inequality

$$|I_2| \leq C h_{\text{ext}}^2 |\omega|^{\frac{1}{2}}$$

and using (6.4) and (6.8) we conclude that I_2 is also $o(1)$. □

Lemma 10. *Using the same notation as in the previous Lemma,*

$$\operatorname{curl} j_1 + h_1 - 2\pi \sum_{i=1}^{k} d_i \delta a_i \tag{6.14}$$

tends to 0 *as* $\varepsilon \to 0$ *in the sense of distributions.*

Proof. It follows from (5.16) that $\operatorname{curl} j + h - 2\pi \sum_{i=1}^{k} d_i \delta a_i$ tends to 0 in the sense of distributions. But from the definition of h_0, j_0 it holds that $\operatorname{curl} j_0 + h_0 = -\Delta h_0 + h_0 = 0$. Therefore $\operatorname{curl} j + h = \operatorname{curl} j_1 + h_1$ and the Lemma is proved. □

We may now deduce

Proposition 14. *Assuming* (6.4), (6.5), (6.6) *are satisfied,*

$$J(u, A) \geq h_{ext}^2 J_0 + \pi \sum_{i=1}^{k} |d_i| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right) + 2\pi h_{ext} \sum_{i=1}^{k} d_i (h_0 - 1)(a_i) + o(1).$$

Proof. Since $j_0 = -\nabla^{\perp} h_0 = -\nabla^{\perp}(h_0 - 1)$ and $(h_0 - 1)$ vanishes on $\partial\Omega$, integration by parts yields

$$\int_{\Omega} j_1 \cdot j_0 + h_1(h_0 - 1) = \int_{\Omega} - \operatorname{curl} j_1 (h_0 - 1) + h_1(h_0 - 1).$$

Thus from Lemma 10

$$\int_{\Omega} j_1 \cdot j_0 + h_1(h_0 - 1) = 2\pi h_{\text{ext}} \sum_{i=1}^{k} d_i (h_0 - 1)(a_i) + o(1).$$

Replacing in (6.13), then in (6.11) and adding (6.9) yields the results. □

6.3 Value of H_{c1}

We now give a meaning to and compute the value of H_{c1} as $\varepsilon \to 0$.

6.3.1 Lower bound

Assume (6.4) is satisfied, and consider a family $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that for each $\varepsilon > 0$ the configuration $(u_\varepsilon, A_\varepsilon)$ is a minimizer of J_ε. Then (6.5) is satisfied from (6.2) and since a minimizer satisfies (1.7), Proposition 8 implies that (6.6) is satisfied also. Therefore Proposition 14 applies and

$$J(u, A) \geq h_{\text{ext}}^2 J_0 + \pi \sum_{i=1}^{k} |d_i| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right) + 2\pi h_{\text{ext}} \sum_{i=1}^{k} d_i (h_0 - 1)(a_i) + o(1),$$

where we have used the same notations as in the previous section. We deduce

Theorem 8. *Assume $h_{ext}(\varepsilon) = \lambda |\log \varepsilon|$. If*

$$\lambda < \frac{1}{2 \max_\Omega |h_0 - 1|}$$

then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$

$$\sum_{i=1}^{k} |d_i| = 0.$$

The meaning of this is that when h_{ext} is below a critical value

$$H_{c1-} \approx \frac{|\log \varepsilon|}{2 \max_\Omega |h_0 - 1|}, \tag{6.15}$$

the minimizers of J_ε have no vortices in the sense of Theorem 7. We will sketch in the next section a proof of the converse statement.

Proof. Using Proposition 14

$$J(u, A) \geq h_{\text{ext}}^2 J_0 + \pi \sum_{i=1}^{k} |d_i| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right) + 2\pi h_{\text{ext}} \sum_{i=1}^{k} d_i (h_0 - 1)(a_i) + o(1)$$

but since (u, A) is a minimizer $J(u, A) \leq J(1, h_{\text{ext}} \nabla^\perp h_0) = h_{\text{ext}}^2 J_0$ therefore

$$0 \geq \pi \sum_{i=1}^{k} |d_i| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right) + 2\pi h_{\text{ext}} \sum_{i=1}^{k} d_i (h_0 - 1)(a_i) + o(1).$$

Let

$$\delta = 1 - 2\lambda \max_\Omega |h_0 - 1|.$$

Then $\delta > 0$ and $|\lambda (h_0 - 1)(x)| \leq 1/2 - \delta$ for any $x \in \Omega$. replacing above yields

$$o(1) \geq \pi \sum_{i=1}^{k} |d_i| \left(|\log \varepsilon| - C \log |\log \varepsilon|\right) - \pi (1 - \delta) |\log \varepsilon| \sum_{i=1}^{k} |d_i|$$

and then

$$o(1) \geq \pi \left(\delta |\log \varepsilon| - C \log |\log \varepsilon|\right) \sum_{i=1}^{k} |d_i|,$$

which proves the Theorem. □

6.3.2 Upper bound

The converse to the above result holds, namely

Theorem 9. *Assume* $h_{ext}(\varepsilon) = \lambda|\log\varepsilon|$. *If*

$$\lambda > \frac{1}{2\max_\Omega |h_0 - 1|}$$

then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$

$$\sum_{i=1}^{k} |d_i| \neq 0.$$

The meaning is that when h_{ext} is above a critical value

$$H_{c1+} \approx \frac{|\log\varepsilon|}{2\max_\Omega |h_0 - 1|}, \tag{6.16}$$

the minimizers of J_ε have vortices in the sense of Theorem 7.

Proof. The proof is by contradiction. If for a given λ minimizers have no vortices, then from (14) the minimal energy is greater then $h_{\text{ext}}^2 J_0$, up to a $o(1)$. Thus it suffices to construct a configuration (u, A) such that if

$$\lambda > \frac{1}{2\max_\Omega |h_0 - 1|}$$

then $J(u, A) < h_{\text{ext}}^2 J_0 - 1$. Of course this configuration must have at least one vortex.

The construction is similar to that of the approximate vortex in $\mathbb{R}^2$. Let $x_0 \in \Omega$ be a point where $\max_\Omega |h_0 - 1|$ is acheived. We define h to solve $-\Delta h + h = 2\pi\delta_{x_0}$ in Ω and $h = h_{\text{ext}}$ on $\partial\Omega$, and take A to be any solution of $\operatorname{curl} A = h$. It remains to define $u = \rho e^{i\varphi}$. We let $\rho(x) = 0$ for $|x - x_0| < \varepsilon$. For $|x - x_0| > 2\varepsilon$ we let $\rho(x) = 1$ and in $B(x_0, 2\varepsilon) \setminus B(x_0, \varepsilon)$ we interploate in the obvious way. Finally we choose φ to solve $-\nabla^\perp h = \nabla\varphi - A$ in $\Omega \setminus B(x_0, \varepsilon)$. Note that φ will only be defined modulo 2π.

It is not difficult to check that $J_{B(x_0,2\varepsilon)}(u, A)$ is bounded independently of ε. In $\widetilde{\Omega} = \Omega \setminus B(x_0, 2\varepsilon)$, using $\rho(x) = 1$ and $-\nabla^\perp h = \nabla\varphi - A$, the Ginzburg-Landau energy rewrites as

$$J_{\widetilde{\Omega}}(u, A) = \frac{1}{2}\|h - h_{\text{ext}}\|^2_{H^1(\widetilde{\Omega})}.$$

It remains to estimate this. Writing $h = h_{\text{ext}} h_0 + h_1$, the function h_1 vanishes on $\partial\Omega$ and solves $-\Delta h_1 + h_1 = 2\pi\delta_{x_0}$ in Ω. It is well known that

$$I_1 = \frac{1}{2}\|h_1\|^2_{H^1(\widetilde{\Omega})} \leq \pi|\log\varepsilon| + C$$

On the other hand

$$I_2 = \frac{1}{2}\|h_{\text{ext}} h_0 - h_{\text{ext}}\|^2_{H^1(\widetilde{\Omega})} \leq h_{\text{ext}}^2 J_0$$

and

$$I_3 = h_{\text{ext}}\langle h_0 - 1, h_1\rangle_{H^1(\widetilde{\Omega})} = h_{\text{ext}}\langle h_0 - 1, h_1\rangle_{H^1(\Omega)} + o(1) = 2\pi h_{\text{ext}}(h_0 - 1)(x_0) + o(1).$$

Adding up we find

$$J(u, A) \leq h_{\text{ext}}^2 J_0 + 2\pi h_{\text{ext}}(h_0 - 1)(x_0) + \pi|\log\varepsilon| + C,$$

note that $h_0 - 1$ is a negative function in Ω. If $h_{\text{ext}} = \lambda|\log\varepsilon|$ and

$$\lambda > \frac{1}{2\max_\Omega |h_0 - 1|} = \frac{1}{2|h_0(x_0) - 1|},$$

we deduce that there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then

$$J(u, A) \leq h_{\text{ext}}^2 J_0 - 1,$$

the Theorem is proved. □

6.4 Conclusion

Theorem 7 gives us a way to define the vortices of a configuration (u, A) which satisfies an apriori bound of the Ginzburg-Landau energy. Then Proposition 14 gives us a lower bound for the Ginzburg-Landau energy in terms of these vortices. This was sufficient to compute the value of h_{ext} for which the vortices appear, but much more could be done. In [35] for instance, using very similar tools to what is presented in these lectures we were able to describe the vortices of minimizers of the Ginzburg-Landau functional as $\varepsilon \to 0$, for different values of $\lambda = h_{\text{ext}}/|\log \varepsilon|$.

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