OPTIMAL INVESTMENT AND HEDGING

UNDER MARKET IMPERFECTIONS

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I. STOCHASTIC CONTROL AND UTILITY PRICING

- Conditional expectations and linear parabolic PDE's
- Standard formulation of stochastic control problems
- Dynamic programming principle and HJB equation
- Verification theorem
- Application 1 : Merton's optimal investment problem
- Application 2 : The Black-Scholes theory in complete markets
- Application 3 : Utility pricing in a stochastic volatility model
- Viscosity solutions of 2nd order PDE's

1. CONDITIONAL EXPECTATIONS AND LINEAR PARABOLIC PDE's

Consider the function :

$$V(t,x) \;:=\; \mathbb{E}_{t,x}\left[\int_t^T f(X_u)\beta(t,u)du + \beta(t,T)g(X_T)\right]$$

where

$$dX_t \;=\; \mu(X_t)dt + \sigma(X_t)dW_t\,, \qquad \beta(t,u) \;:=\; e^{-\int_t^u k(X_v)dv}$$

and

$$\mu \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}^n\,, \qquad \sigma \;:\; \mathbb{R}^n \longrightarrow \mathcal{S}^n_{\mathbb{R}}\,,$$

$$f,\; g,\; k \;:\quad \mathbb{R}^n \longrightarrow \mathbb{R}$$

Second order PDE :

$$\text{(E)} \quad \frac{\partial v}{\partial t}(t,x) + F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) = 0, \; t<T, \; x \in \mathcal{O} \subset \mathbb{R}^n$$

• (E) is parabolic if $F(x,r,p,A)$ is non-increasing in A

• (E) is linear if $F(x,r,p,A)$ is linear in (r,p,A)

• v is a classical super-solution (resp. subsolution) of (E) is $v \in C^{1,2}$ and $\frac{\partial v}{\partial t} + F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) \geq 0$ (resp. ≤ 0) on $[0,t) \times \mathbb{R}^n$

Maximum Principle. *Let $\mathcal{O}$ bounded and $F(t,x,r,p,A)$ parabolic strictly increasing in r. Let u (resp. v) be a classical subsolution (resp. super-solution) of (E), with $u \leq v$ on $\partial\{(0,T)\times\mathcal{O}\}$. Then $u \leq v$ on $[0,T]\times\overline{\mathcal{O}}$.*

Dynkin operator

$$\mathcal{L}V(t,x) \;:=\; V_t(t,x) + \mu(x)\cdot DV(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x)D^2V(t,x)\right]$$

$\Longrightarrow$ Tower property : for any $h > 0$

$$\beta(0,t)V(t,x) \;=\; \mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(0,u)f(X_u)du + \beta(0,t+h)V(t+h,X_{t+h})\right]$$

$\Longrightarrow$ if V is smooth, then it follows from Itô's lemma

$$0 \;=\; \frac{1}{h}\mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(t,u)\left(kV-\mathcal{L}V-f\right)(u,X_u)du + \int_t^T DV(u,X_u)\cdot\sigma(X_u)dW\right.$$

$$\;=\; \frac{1}{h}\mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(t,u)\left\{k(X_u)V(u,X_u)-\mathcal{L}V(u,X_u)-f(X_u)\right\}du\right] \qquad \textbf{!}$$

send h to zero $\Longrightarrow$ V solves the parabolic linear PDE

$$-\mathcal{L}V(t,x) + k(x)V(t,x) - f(x) \;=\; 0$$

Feynman-Kac representation formula

Cauchy problem

$$-\mathcal{L}v(t,x) + k(x)v(t,x) - f(x) \;=\; 0 \;\text{ and }\; v(T,x) \;=\; g(x)$$

Theorem *Let v be a classical solution of the above Cauchy problem with $|v(t,x)| \le C(1+|x|^p)$. Then*

$$v(t,x) \;=\; V(t,x) \;=\; \mathbb{E}_{t,x}\left[\int_t^T f(X_u)\beta(t,u)du + \beta(t,T)g(X_T)\right]$$

- Uniqueness
- Important implication for numerical approximation

Cauchy problem can be solved

by means of Monte Carlo method

2. STANDARD STOCHASTIC CONTROL PROBLEMS

- *Control process* $\nu \in \mathcal{U}_0$

$$\nu_t \; \mathcal{F}_t - \text{measurable r.v. with values in } \; U \; \subset \; \mathbb{R}^k$$

- *Controlled process* For $\nu \in \mathcal{U}_0$, define X^ν by

$$\text{EDS}(\nu) \quad dX_t^\nu \; = \; b(X_t^\nu, \nu_t)dt + \sigma(X_t^\nu, \nu_t)dW_t \;\text{ and }\; X_0^\nu \text{ given}$$

where

$$b \; : \; \mathbb{R}^n \times U \; \longrightarrow \; \mathbb{R}^n, \quad \sigma \; : \; \mathbb{R}^n \times U \; \longrightarrow \; \mathcal{M}_{\mathbb{R}}^{n,d} \quad \text{Lip in } x \text{ unif. in } u$$

- *Admissible control process* $\nu \in \mathcal{U}$ if

EDS(ν) has a unique solution in L^2 for every initial data $X_0 = x$

2.1. REWARD CHARACTERISTICS

$$f, k \ : \ \mathbb{R}^n \times U \longrightarrow \mathbb{R} \ \text{ and } \ g \ : \ \mathbb{R}^n \longrightarrow \mathbb{R}$$

with

$$k \geq 0 \ \text{ and } \ |f(t,x,u)| + |g(x)| \ \leq \ C(1 + |x|^2)$$

- f : continuous reward rate

- g : terminal reward

- k : discount rate

2.2. STOCHASTIC CONTROL PROBLEM

$$V(t,x) \;:=\; \sup_{\nu\in\mathcal{U}} J(t,x,\nu)$$

where

$$J(t,x,\nu) \;:=\; \mathbb{E}_{t,x}\left[\int_t^T \beta^\nu(t,s) f\left(X_s^\nu,\nu_s\right) ds + \beta^\nu(t,T) g\left(X_T^\nu\right)\right]$$

with the discount factor

$$\beta^\nu(t,s) \;:=\; e^{-\int_t^s k(X_r^\nu,\nu_r)dr}$$

Goal : characterize the local behavior of V by means of

the *Hamilton-Jacobi-Bellman* equation

2.3. SOME VOCABULARY

- $\hat{\nu}$ is an *optimal control* if

$$\hat{\nu} \in \mathcal{U} \quad \text{and} \quad V(t,x) \ = \ J(t,x,\hat{\nu})$$

- $\nu \in \mathcal{U}$ is a *feedback control*

$$\nu \text{ is adapted to } \mathbb{F}^X$$

- ν is a *Markov control* if

$$\nu_s \ = \ u(s, X_s) \quad \text{for some measurable function} \quad u$$

- ν is an *open-loop control* if

$$\nu \quad \text{is deterministic}$$

2.4. DYNAMIC PROGRAMMING PRINCIPLE

Theorem *For any stopping time* τ *with values in* $[t, T]$ *:*

$$V(t,x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^\tau \beta^\nu(t,s) f\left(s, X_s^\nu, \nu_s\right) ds + \beta^\nu(t,\tau) V\left(\tau, X_\tau^\nu\right)\right]$$

- Basic tool of stochastic control / compare with tower property
- *Main ingredient* : concatenation of control processes
- In finite discrete time

$$V(t,x) = \sup_{u \in U} \mathbb{E}_{t,x}\left[f\left(X_t^\nu, u\right) + e^{-k(X_t^\nu, \nu_t)} V\left(t+1, X_{t+1}^\nu\right)\right]$$

$\Longrightarrow$ Reduction to a (backward) sequence of finite-dimensional optimization problems

2.5. REDUCTION TO MAYER FORM ($f = k \equiv 0$)

Consider new controlled processes (Y, Z) :

$$dY_s^\nu \;=\; Z_s^\nu \, f(X_s^\nu, \nu_s)ds \;\text{ and }\; dZ_s^\nu \;=\; -Z_s^\nu \, k(X_s^\nu, \nu_s)ds$$

$\Longrightarrow$ Augmented controlled process

$$\bar{X}^\nu \;:=\; (X^\nu, Y^\nu, Z^\nu)$$

Then $V(t, x) \;=\; \bar{V}(t, x, 0, 1)$, where

$$\bar{V}(t, \bar{x}) \;:=\; \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[\bar{g}\left(\bar{X}_T^\nu\right)\right] \;\text{ and }\; \bar{g}(x, y, z) \;:=\; y + g(x)z$$

2.6. HAMILTON-JACOBI-BELLMAN EQUATION

Denote

$$\mathcal{L}^u v(t,x) \;:=\; v_t(t,x) + b(x,u)\cdot Dv(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x,u)D^2v(t,x)\right]$$

$$H(x,r,p,A) \;:=\; \sup_{u\in U}\left\{-k(x,u)r + b(x,u)\cdot p + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(x,u)A] + f(x,u)\right\}$$

Proposition *If* $V \in C^{1,2}([0,T),\mathbb{R}^n)$ *:*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(x, V(t,x), DV(t,x), D^2V(t,x)\right) \;\geq\; 0$$

i.e. V *is a super-solution of the associated HJB equation*

Proof of super-solution property. $(t,x) \in [0,T) \times \mathbb{R}^n$, $u \in U$ fixed, constant control $\nu_s = u$, controlled process X^u, and

$$\tau_h \;:=\; (t+h) \wedge \inf\{s > t \;:\; |X_s^u - x| \geq 1\}$$

Dynamic programming and Itô's lemma :

$$\begin{aligned}
0 \;\leq\; & \frac{1}{h}\,\mathbb{E}_{t,x}\left[\beta(0,t)V(t,x) - \beta(0,\tau_h)V(\tau_h, X_{\tau_h}) - \int_t^{\tau_h} \beta(0,r) f(r, X_r, \nu_r) dr\right] \\
=\; & -\frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r)(-kV + \mathcal{L}^u V + f)(r, X_r, u) dr\right] \\
& \qquad\qquad - \frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r) DV(r, X_r)^T \sigma(X_r, u) dW_r\right] \\
=\; & -\frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r)(-kV + \mathcal{L}^u V + f)(r, X_r, u) dr\right]
\end{aligned}$$

Finally, send h to zero, and use the dominated convergence theorem

Proposition *If $V \in C^{1,2}([0,T), \mathbb{R}^n)$, and H is continuous, then :*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(x, V(t,x), DV(t,x), D^2V(t,x)\right) \ = \ 0$$

$\Longrightarrow$ Proof... more technical

In order to complete the characterization of V :

(i) Terminal condition

(ii) Uniqueness result

2.7. VERIFICATION RESULT

Theorem $v \in C^{1,2}([0,T), \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *with* $|v(t,x)| \leq C(1+|x|^2)$

(i) *If* $v(T,\cdot) \geq g$ *and* $-v_t(t,x) - H\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) \geq 0$.

Then $v \geq V$

(ii) *Assume further that*

• $v(T,.) = g$ *and* $0 = \mathcal{L}^{\hat{u}(t,x)}v(t,x) - k\left(x,\hat{u}(t,x)\right)v(t,x) + f\left(x,\hat{u}(t,x)\right)$ *for some measurable function* $\hat{u}$

• *there is a unique solution for the SDE*

$$dX_s = b\left(X_s,\hat{u}(s,X_s)\right)ds + \sigma\left(X_s,\hat{u}(s,X_s)\right)dW_s \quad \text{for any} \quad X_0 = x$$

• $\hat{\nu} \in \mathcal{U}$, *where* $\hat{\nu}_s := \hat{u}(s,X_s)$

Then $v = V$, *et* $\hat{\nu}$ *is a (Markov) optimal control*

Sketch of the proof

(i) Let $\nu \in \mathcal{U}$, $X = X^\nu$, $X_t = x \Longrightarrow$ Itô's lemma :

$$\begin{aligned} v(t,x) &= \beta(t,T) v\left(T, X_T^\nu\right) \\ &\quad - \int_t^T \beta^\nu(t,r)(-kv + \mathcal{L}^{\nu(r)} v)(r, X_r^\nu) dr \\ &\quad - \int_t^T \beta^\nu(t,r) Dv(r, X_r^\nu) \cdot \sigma(r, X_r^\nu, \nu_r) dW_r \end{aligned}$$

Since $kv - \mathcal{L}^u v - f(\cdot, u) \geq -v_t - H(\cdot, v, Dv, D^2 v) \geq 0$:

$$\begin{aligned} v(t,x) &\geq \mathbb{E}_{t,x}\left[\beta^\nu(t,T) v\left(T, X_T^\nu\right) + \int_t^T \beta^\nu(t,r) f(X_r^\nu, \nu_r) dr\right] \\ &\geq \mathbb{E}_{t,x}\left[\beta^\nu(t,T) g\left(X_T^\nu\right) + \int_t^T \beta^\nu(t,r) f(X_r^\nu, \nu_r) dr\right] \end{aligned}$$

(ii) inequalities are in fact equalities with the control $\hat{\nu}$

3.1. APPLICATION 1 : MERTON OPTIMAL INVESTMENT

• Non-risky asset (bank account) $\tilde{S}^0_t = e^{rt}$, $t \geq 0$

• 1 risky asset (to simplify) $d\tilde{S}_t = \tilde{S}_t\,(\tilde{\mu}\,dt + \sigma\,dW_t)$

• Discounted prices (change of numéraire) $S^0 := \tilde{S}^0/\tilde{S}^0$ and $S := \tilde{S}/\tilde{S}^0$:

$$S^0 \equiv 1 \quad \text{and} \quad dS_t = S_t\,(\mu\,dt + \sigma\,dW_t)$$

where $\mu := \tilde{\mu} - r$

• By Itô's lemma : $S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$

(Log-normal distribution)

• **Portfolio strategy :** wealth X_t allocated into

$\longrightarrow$ $\pi_t X_t$: investment amount in risky asset S

$\longrightarrow$ $X_t - \pi_t X_t$: investment amount in the bank S^0

$\Longrightarrow$ Under self-financing condition :

$$dX_t \;=\; \frac{\pi_t X_t}{S_t}\, dS_t \;=\; X_t\, \pi_t\, (\mu dt + \sigma dW_t)$$

Given an initial capital $X_0 = x$, we denote :

$$\begin{aligned} X_t^{x,\pi} &:= x + \int_0^t X_u^{x,\pi} \pi_u\, (\mu du + \sigma dW_u) \\ &= x \exp\left[\int_0^t \left(\pi_u \mu - \frac{1}{2}|\pi_u|^2 \sigma^2\right) du + \pi_u \sigma dW_u\right] \end{aligned}$$

$\mathcal{A}$ - set of admissible strategies : $\int_0^T |\pi_u|^2 du \;<\; \infty$ a.s.

• **Optimal investment problem :** Let $U(x) := \frac{x^p}{p}$, $p < 1$, and

$$V(0,x) \;:=\; \sup_{\pi \in \mathcal{A}} \; \mathbb{E}\left[U\left(X_T^{x,\pi}\right)\right]$$

and observe immediately that

$$V(0,x) \;=\; \sup_{\pi \in \mathcal{A}} \; \mathbb{E}\left[U\left(xX_T^{1,\pi}\right)\right] \;=\; x^p\, V(0,1)$$

which extends obviously to the dynamic version of the problem

$$V(t,x) \;=\; x^p\, V(t,1)$$

- HJB equation $$-\frac{\partial V}{\partial t} - \sup_{\pi\in\mathbb{R}} \left\{\pi\mu x V_x + \frac{1}{2}\pi^2\sigma^2 x^2 V_{xx}\right\} = 0$$
- Terminal condition $$V(T,x) = \frac{x^p}{p}$$

- Find $v(t,x) := x^p h(t)$ solution of above PDE problem :

$$0 \;=\; h' + ph \sup_{\pi\in\mathbb{R}} \left\{\pi\mu + \frac{1}{2}(p-1)\pi^2\sigma^2\right\} \;=\; h' + \frac{1}{2}\,\frac{p}{1-p}\,\frac{\mu}{\sigma^2}\,h$$

$\Longrightarrow$ Explicit unique solution satisfying $h(T) = p^{-1}$:

$$h(t) \;:=\; \exp\left[\frac{1}{2}\,\frac{p}{1-p}\,\frac{\mu}{\sigma^2}\,(T-t)\right]$$

- Observe that optimal portfolio strategy

$$\pi_t^* \;:=\; \frac{\mu}{(1-p)\sigma^2} = \frac{\tilde{\mu}-r}{(1-p)\sigma^2}$$

By the verification theorem, we have $V = v = x^p\, h(t)$

3.2. APPLICATION 2 : BLACK-SCHOLES VALUATION

(utlity indifference approach)

In the setting of Application 1, consider two stochastic control problems

$$V^g(0, x, S_0) \;:=\; \sup_{\pi \in \mathcal{A}} E\left[U\left(X_T^{x,\pi} - g(S_T)\right)\right]$$

$$V^0(0, x, S_0) \;:=\; \sup_{\pi \in \mathcal{A}} E\left[U\left(X_T^{x,\pi} - 0\right)\right]$$

where $g(S_T)$ is a contingent claim (random payment at time T)

- Utility indifference valuation rule

$$p^0(0, x, s) \;:=\; \inf\left\{p \;:\; V^g(0, x+p, s) \;\geq\; V^0(0, x, s)\right\}$$

- Observe that V^0 independent of S_0 :

$$V^0(0,x,S_0) := \sup_{\theta\in\mathcal{A}} E\left[U\left(X_T^{x,\theta}\right)\right]$$
$$= \sup_{\theta\in\mathcal{A}} E\left[U\left(x+\int_0^T X_u^{x,\pi}\pi_u\left(\mu du+\sigma dW_u\right)\right)\right]$$
$$= V^0(0,x)$$

- V^g is continuous ($\Longleftarrow$ concave) strictly increasing in $x \Longrightarrow$

$$V^g\left(0, x+p^0(0,x,s), s\right) = V^0(0,x)$$

- Let π^g be the solution of V^g, then

$$\pi^g - \pi^0 \text{ is the hedging strategy for } g(S_T)$$

• **HJB equation for** V^g

$$-\sup_{\pi\in\mathbb{R}} \mathcal{L}^\pi V^g(t,x,s) = 0 \quad \text{and} \quad V^g(T,x,s) = U(x-g(s)),$$

where

$$\mathcal{L}^\pi V^g := V_t^g + \mu s V_s^g + \frac{1}{2}\sigma^2 s^2 V_{ss}^g + \mu\pi x V_x^g + \frac{1}{2}\sigma^2\pi^2 x^2 V_{xx}^g + \sigma^2 s\pi x V_{xs}^g$$

• **HJB equation for** $V^0(t,x)$

$$-\sup_{\pi\in\mathbb{R}} \mathcal{L}_0^\pi V^0(t,x) = 0 \quad \text{and} \quad V^0(T,x) = U(x),$$

where

$$\mathcal{L}_0^\pi V^0 := V_t^0 + \mu\pi x V_x^0 + \frac{1}{2}\sigma^2\pi^2 x^2 V_{xx}^0$$

- V^g is concave in $x \Longrightarrow$

$$
\begin{aligned}
& \sup_{\pi\in\mathbb{R}} \left[\mu\pi x V_x^g + \frac{1}{2}\sigma^2\pi^2x^2V_{xx}^g + \pi\sigma^2 s x V_{xs}^g\right] \\
= \;& \frac{1}{2}\sigma^2 V_{xx}^g \inf_{\pi\in\mathbb{R}} \left[2\pi x\left(\frac{\mu V_x^g + \sigma^2 s V_{xs}^g}{\sigma^2 V_{xx}^g}\right) + \pi^2x^2\right] \\
= \;& \frac{1}{2}\sigma^2 V_{xx}^g \left\{-\left(\frac{\mu V_x^g + \sigma^2 s V_{xs}^g}{\sigma^2 V_{xx}^g}\right)^2 + \inf_{\pi\in\mathbb{R}}\left(\pi x + \frac{\mu V_x^g + \sigma^2 s V_{xs}^g}{\sigma^2 V_{xx}^g}\right)^2\right\} \\
= \;& -\frac{1}{2}\sigma^2 V_{xx}^g \left(\frac{\mu V_x^g + \sigma^2 s V_{xs}^g}{\sigma^2 V_{xx}^g}\right)^2 \quad \text{with} \quad \pi^g x := -\frac{\mu V_x^g + \sigma^2 s V_{xs}^g}{\sigma^2 V_{xx}^g}
\end{aligned}
$$

- Similarly,

$$
\sup_{\pi\in\mathbb{R}} \left[\mu\pi x V_x^0 + \frac{1}{2}\sigma^2\pi^2x^2V_{xx}^0\right] = -\frac{1}{2}\sigma^2\theta^2 V_{xx}^0 \left(\frac{\mu V_x^0}{\sigma^2 V_{xx}^0}\right)^2
$$

$$
\text{with} \quad \pi^0 x := -\frac{\mu V_x^0}{\sigma^2 V_{xx}^0}
$$

$\Longrightarrow$ HJB equation for V^g :

$$(\mathsf{E}^g) \qquad \begin{cases} -V_t^g - \mu s V_s^g - \frac{1}{2}\sigma^2 s^2 V_{ss}^g + \frac{1}{2}\frac{\left(\mu V_x^g + \sigma^2 s V_{xs}^g\right)^2}{\sigma^2 V_{xx}^g} = 0 \\ V^g(T,x,s) = U(x - g(s)) \end{cases}$$

$\Longrightarrow$ HJB equation for V^0 :

$$\left(\mathsf{E}^0\right) \qquad \begin{cases} -V_t^0 + \frac{1}{2}\frac{\left(\mu V_x^0\right)^2}{\sigma^2 V_{xx}^0} = 0 \\ V^0(T,x) = U(x) \end{cases}$$

Consider the function

$$f(t,x,s) \;:=\; V^0(t, x - p(t,s))$$

$p(t,s)$ is the solution of the Black-Scholes PDE (heat equation)

$$-p_t - \frac{1}{2}\sigma^2 s^2 p_{ss} \;=\; 0 \qquad p(T,.) = g$$

1. If V^0 is smooth, then $f(t,x,s)$ is a classical solution of (E^g)
2. Additional conditions on V^0 allow to apply our Verification result

$$\Longrightarrow \qquad V^g(t,x,s) \;:=\; V^0(t, x - p(t,s))$$

and therefore, the utility indifference price is

$$p^0(t,s) \;=\; p(t,s)$$

With $f(t,x,s) := V^0(t, x - p(t,s))$, we have

$$f_t = V_t^0 - V_x^0 p_t,\ f_x = V_x^0,\ f_{xx} = V_{xx}^0$$

$$f_s = -V_x^0 p_s,\ f_{ss} = -V_x^0 p_{ss} + V_{xx}^0 (p_s)^2,\ f_{xs} = -V_{xx}^0 p_s$$

and we compute that :

$$-f_t - \mu s f_s - \tfrac{1}{2}\sigma^2 s^2 f_{ss} + \frac{1}{2}\sigma^2 f_{xx}\left(\frac{\mu f_x + \sigma^2 s f_{xs}}{\sigma^2 f_{xx}}\right)^2$$

$$= -V_t^0 + \tfrac{1}{2}\sigma^2 V_{xx}^0 \left(\frac{\mu V_x^0}{\sigma^2 V_{xx}^0}\right)^2 + V_x^0\left[p_t + \tfrac{1}{2}\sigma^2 s^2 p_{ss}\right]$$

$$= 0$$

Martingale approach for the Black-Scholes model

• Let x be such that : $\exists\, \pi \in \mathcal{A},\ X_T^{x,\pi} \geq g(S_T)$

$$X_T^{x,\pi} \;=\; x + \int_0^T X_u^{x,\pi}\pi_u(\mu du + \sigma dW_u) \;=\; x + \int_0^T X_u^{x,\pi}\pi_u \sigma d\tilde{W}_u$$

where $\tilde{W}$ is a Brownian motion under $\tilde{P} \sim P$ (Girsanov theorem). Then

$$x \;\geq\; E^{\tilde{P}}\left[X_T^{x,\pi}\right] \;\geq\; E^{\tilde{P}}\left[g(S_T)\right] \;\text{ and }\; V_0 \;\geq\; E^{\tilde{P}}\left[g(S_T)\right]$$

• Representation of the $\tilde{P}-$martingale $M_t := E^{\tilde{P}}\left[g(S_T)|\mathcal{F}_t\right]$:

$$M_t \;=\; M_0 + \int_0^t \phi_u d\tilde{W}_u \;=\; M_0 + \int_0^t M_u \hat{\pi}_u \sigma d\tilde{W}_u \;=\; X_t^{M_0,\hat{\pi}}$$

($g \geq 0$ and $g \not\equiv 0$) Since $X_T^{M_0,\hat{\pi}} = g(S_T)$, we conclude that

$$M_0 \;=\; E^{\tilde{P}}\left[g(S_T)\right] \;\geq\; V_0$$

Observe

• $M_t := E^{\tilde{P}}\left[g(S_T)|\mathcal{F}_t\right] = E^{\tilde{P}}\left[g(S_T)|S_t\right] = p(t, S_t)$

where $p(t, s)$ is the solution of the Black-Scholes PDE (heat equation)

$$-p_t - \frac{1}{2}\sigma^2 s^2 p_{ss} \;=\; 0 \qquad p(T, .) = g$$

Theorem *For* $g \geq 0$ *and* $g \not\equiv 0$*, we have*

$$p(0, S_0) \;=\; \inf\left\{x \;:\; X_T^{x,\pi} \geq g(S_T) \textit{ a.s. for some } \pi \in \mathcal{A}\right\}$$

Moreover, there exists $\widehat{\pi} \in \mathcal{A}$ *such that*

$$X_T^{p(0,S_0),\widehat{\pi}} \;=\; g(S_T) \qquad \textit{(perfect replication)}$$

Complement 1 : direct proof of

utility indifference price = Black-Scholes price

in the context of a *COMPLETE MARKET*

Since $g(S_T) = M_T = X_T^{M_0,\widehat{\pi}}$ and $\mathcal{A}$ is a vector space,

$$\begin{aligned} V^{g}(0,x,S_0) &:= \sup_{\pi\in\mathcal{A}} E\left[U\left(X_T^{x,\pi} - g(S_T)\right)\right] \\ &= \sup_{\pi\in\mathcal{A}} E\left[U\left(X_T^{x,\pi} - X_T^{M_0,\widehat{\pi}}\right)\right] \\ &\stackrel{*}{=} \sup_{\pi\in\mathcal{A}} E\left[U\left(X_T^{x-M_0,\pi-\widehat{\pi}} - 0\right)\right] \\ &= V^{0}(0,x-M_0,S_0) \end{aligned}$$

$\stackrel{*}{=}$: assume for instance $U'(0+) = +\infty$

Complement 2

Connection with Backward SDE's

in the context of *COMPLETE MARKET*

the pair $\left(X_t^{V_0,\widehat{\pi}},\, X_t^{V_0,\widehat{\pi}}\widehat{\pi}_t\sigma\right)_{0,\leq t\leq T}$ is the unique solution of the Backward stochastic differential equation

$$dY_t \;=\; Z_t\, dW_t \;\text{ and }\; Y_T \;=\; g\left(S_T\right)$$

(see the lectures by A. Matoussi in this school)

3.3. APPLICATION 3 : UTILITY INDIFFERENCE PRICING IN INCOMPLETE MARKETS

< Zariphopoulou 99, Finance and Stochastics >

• Non-risky asset $S^0 \equiv 1$ (discounting / change of numéraire)

• 1 risky asset S defined by : $\dfrac{dS_t}{S_t} = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t^1$

Y non-tradable state variable : $dY_t = \eta(Y_t)\,dt + \vartheta_1(Y_t)dW_t^1 + \vartheta_2(Y_t)dW_t^2$

W^1, W^2 are independent Brownian motions

η, ϑ_1, ϑ_2 Lipschitz, μ, σ bounded and $\inf\limits_{s,y} \sigma(s,y)^2 + \vartheta_2(s,y)^2 > 0$

• Wealth process : $X_0^{x,\theta} = x$ and $dX_t^{x,\theta} = \theta_t \,\dfrac{dS_t}{S_t}$

where θ adapted process, $\int_0^T |\theta_t|^2 dt < \infty$ a.s. and $X^{0,\theta} \geq C \longrightarrow \mathcal{A}$

- Let $G = g(Y_T)$ be a contingent claim, $g \; : \; \mathbb{R} \longrightarrow \mathbb{R}$ bounded
- Define two stochastic control problems

$$V^g(0,x,Y_0) \;:=\; \sup_{\theta\in\mathcal{A}} E\left[U\left(X_T^{x,\theta} - g(Y_T)\right)\right]$$
$$V^0(0,x,Y_0) \;:=\; \sup_{\theta\in\mathcal{A}} E\left[U\left(X_T^{x,\theta} - 0\right)\right]$$

(independent of S_0). Indifference valuation rule :

$$p^0(0,x,y) \;:=\; \inf\left\{p \;:\; V^g(0,x+p,y) \;\geq\; V^0(0,x,y)\right\}$$

- V^g continuous ($\Longleftarrow$ concave) strictly increasing in x (if U is) $\Longrightarrow$

$$V^g\left(0, x + p^0(0,x,y), y\right) \;=\; V^0(0,x,y)$$

- Utility function $U(x) = -e^{-\eta x} \Longrightarrow V^g(t,x,y) = e^{-\eta x}\, V^g(t,0,y)$

HJB equation for V^g

$$-\sup_{\theta\in\mathbb{R}} \mathcal{L}^\theta V^g(t,x,y) \;=\; 0 \;\;\text{and}\;\; V^g(T,x,y) \;=\; U(x-g(y))$$

where $\mathcal{L}^\theta V^g := V^g_t + \eta V^g_y + \frac{1}{2}(\vartheta_1^2+\vartheta_2^2)V^g_{yy} + \mu\theta V^g_x + \frac{1}{2}\sigma^2\theta^2 V^g_{xx} + \theta\sigma\vartheta_1 V^g_{xy}$

- Recalling that $V^g(t,x,y) = -e^{-\eta x}F^g(t,y)$, with $F^g(t,y) := -V^g(t,0,y)$

$$-\sup_{\theta\in\mathbb{R}} \mathcal{G}^\theta F^g(t,y) \;=\; 0 \;\;\text{and}\;\; F^g(T,y) = -U(-g(y)) \;=\; e^{\eta g(y)}$$

where $\mathcal{G}^\theta F^g := F^g_t + \eta F^g_y + \frac{1}{2}(\vartheta_1^2+\vartheta_2^2)F^g_{yy} + \frac{1}{2}\eta^2\sigma^2\theta^2 F^g - \eta\theta\left(\mu F^g + \sigma\vartheta_1 F^g_y\right)$

- $F^g \geq 0 \Longrightarrow \mathcal{G}^\theta F^g$ is convex in $\theta \Longrightarrow$

$$\mathcal{G}^0 F^g - \frac{\left(\mu F^g + \sigma\vartheta_1 F^g_y\right)^2}{2\sigma^2 F^g} \;=\; 0 \;\;\text{and}\;\; F^g(T,y) \;=\; e^{\eta g(y)}$$

- **Optimal portfolio** $\quad \hat{\theta}^g \;:=\; \hat{\theta}^g \;:=\; \dfrac{\mu F^g + \sigma\vartheta_1 F^g_y}{\eta\sigma^2 F^g}$

Representation by means of BSDE's

$\phi(t,y) := \ln[F^g(t,y)] := \ln[-V^g(t,0,y)]$ solves

$$\begin{aligned} \phi_t + \left(\eta - \tfrac{\mu}{\sigma}\vartheta_1\right)\phi_y + \tfrac{1}{2}(\vartheta_1^2 + \vartheta_2^2)\phi_{yy} - \tfrac{\mu}{2\sigma^2} - \tfrac{\vartheta_1^2}{2}\,\phi_y^2 &= 0 \\ \phi(T,y) &= \eta g(y) \end{aligned}$$

$\Longrightarrow$ semilinear PDE

Then $\phi(0,x) = Y_0$, where (Y,Z) is the unique solution the BSDE

$$dY_t = \left(-\frac{\mu(X_t)}{2\sigma^2(X_t)} - \frac{\vartheta_1^2(X_t)}{2\sqrt{\vartheta_1^2 + \vartheta_2^2}(X_t)}\, Z_t^2 \right) dt + Z_t\, dB_t\,, \quad Y_T = \eta g(X_T)$$

B is a Brownaian motion, and X is the unique solution of the SDE

$$X_0 = x\,, \qquad dX_t = \left(\eta - \frac{\mu}{\sigma}\vartheta_1\right)(X_t)\,dt + \sqrt{\vartheta_1^2 + \vartheta_2^2}(X_t)\,dB_t$$

Assume further that $\dfrac{\vartheta_1^2(y)}{\vartheta_1^2(y)+\vartheta_2^2(y)} =: \rho^2$: constant

• Change of variable : $f^g(t,y) := F^g(t,y)^{1/(1-\rho^2)} \Longrightarrow$

$$\begin{cases} f_t^g + (\eta - \rho\mu\vartheta^2) f_y^g + \frac{1}{2}\vartheta^2 f_{yy}^g - \frac{\mu^2(1-\rho^2)}{\sigma^2} f^g = 0 \\ f^g(T,y) = e^{\eta(1-\rho^2)g(y)} \end{cases}$$

$\vartheta := \sqrt{\vartheta_1^2 + \vartheta_2^2}$

• Feynman-Kac formula $\Longrightarrow$

$$f^g(t,y) = E\left[e^{(1-\rho^2)\left(\eta g(\tilde{Y}) - \frac{1}{2}\int_t^T \lambda^2(\tilde{Y}_u)du\right)}\right]$$

where $\lambda := \dfrac{\mu}{\sigma}$ (risk premium of S),

$$d\tilde{Y} = \left[\eta - \rho\mu\vartheta^2(\tilde{Y})\right] dt + \vartheta(\tilde{Y})dB_t \ \text{ and } \ B \text{ is a BM}$$

• **Indifference pricing rule** $V^g\left(t, x + p^0(t,x,y), y\right) = V^0(t,x,y)$

$$e^{-\eta p^0(t,x,y)} f^g(t,y)^{1/(1-\rho^2)} = f^0(t,y)^{1/(1-\rho^2)}$$

i.e.

$$p^0(t,y) = \frac{1}{\eta(1-\rho^2)} \ln\left(\frac{f^g(t,y)}{f^0(t,y)}\right) = \frac{1}{\eta(1-\rho^2)} \ln E^Q\left[e^{\eta(1-\rho^2)\, g(\tilde{Y})}\right]$$

where $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \alpha \exp\left[-\frac{1}{2}(1-\rho^2)\int_0^t \lambda^2(\tilde{Y}_u)du\right]$

• **If λ is constant** : $p^0(t,y) = \frac{1}{\eta(1-\rho^2)} \ln E\left[e^{(1-\rho^2)\eta g(\tilde{Y})}\right]$

• **Optimal hedging portfolio** :

$$\widehat{\theta}^g(t,y) - \widehat{\theta}^0(t,y) = \frac{\rho}{\eta\sigma(y)}\vartheta(y)\frac{\partial}{\partial y}\ln\left(\frac{f^g}{f^0}\right)(t,y) = \frac{\rho(1-\rho^2)}{\sigma(y)}\vartheta(y)\, p^0_y(t,y)$$

4. ON THE REGULARITY OF THE VALUE FUNCTION

$f = k \equiv 0$ (Mayer's formulation)

Proposition (i) g *Lipschitz, then* $V(t, \cdot)$ *is Lipschitz-continuous*

(ii) U *bounded, then* $V(\cdot, x)$ *is* $(1/2)-$*Hölder-continuous*

Example. Let $U = \mathbb{R}, \mathcal{U} := \{$bounded predictable processes valued in$U\}$,

$$dX_t^\nu = \nu_t \, dW_t \quad \text{and} \quad V(t,x) := \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[g\left(X_T^\nu\right)\right]$$

where $g^{conc} < \infty$. Then

$$V(t,x) = g^{conc}(x) \qquad g^{conc} \text{ is the concave envelope of } g$$

V not continuous at $t = T$ and not C^1 in x, in general.

5. VISCOSITY SOLUTIONS

Consider the elliptic PDE

(E) $\quad F\left(z, v(z), Dv(z), D^2v(z)\right) = 0$ for $z \in \mathcal{O}$ open subset of $\mathbb{R}^d$

($F(z,r,p,A)$ non-increasing in A)

• $v : \mathcal{O} \longrightarrow \mathbb{R}$ l.s.c. is a viscosity super-solution of (E) if, for every $(z_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$:

$$(v-\varphi)(z_0) = \min_{\mathcal{O}} (v-\varphi) \implies F\left(z_0, v(z_0), D\varphi(z_0), D^2\varphi(z_0)\right) \geq 0$$

• $v : \mathcal{O} \longrightarrow \mathbb{R}$ u.s.c. is a viscosity sub-solution of (E) if, for every $(z_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$:

$$(v-\varphi)(z_0) = \max_{\mathcal{O}} (v-\varphi) \implies F\left(z_0, v(z_0), D\varphi(z_0), D^2\varphi(z_0)\right) \leq 0$$

Semi-continuous envelopes :

$$v_*(z) \ := \ \liminf_{z' \to z} \ v(z') \quad \text{and} \quad v_*(z) \ := \ \limsup_{z' \to z} \ v(z')$$

finite for locally bounded $v \ : \ \mathbb{R}^d \longrightarrow \mathbb{R}$

Proposition (i) *If V is locally bounded, then*

$$-\frac{\partial V_*}{\partial t}(t,x) - H\left(x, V_*(t,x), DV_*(t,x), D^2 V_*(t,x)\right) \ \geq \ 0$$

i.e. V_* is a super-solution of the associated HJB equation

(ii) *If in addition H is continuous, then*

$$-\frac{\partial V^*}{\partial t}(t,x) - H\left(x, V^*(t,x), DV^*(t,x), D^2 V^*(t,x)\right) \ \leq \ 0$$

i.e. V^* is a sub-solution of the associated HJB equation

UNIQUE CHARACTERIZATION AND CONTINUITY

• Boundary condition (Recall our example) :

$V_*(T,x)$ and $V^*(T,x)$ might not be given by the natural BC $g(x)$

• If we can prove that $V_*(T,x) \geq V^*(T,x)$ and that Maximum principle in the viscosity sense holds, then :

$$V_* = V^* \text{ on } [0,T] \times \mathbb{R}^n$$

$\Longrightarrow V$ is the unique continuous viscosity solution in a certain class.

Users' guide : Crandal, Ishii and Lions 92

II. SUPER-HEDGING UNDER PORTFOLIO CONSTRAINTS

- Problem formulation
- Dual formulation
- Geometric dynamic programming and HJB equation
- Boundary condition : *face-lifting*
- Explicit solution in the Black-Scholes model

1.1. PROBLEM FORMULATION : the financial market

- 1 non-risky asset $S^0 \equiv 1$ ($r = 0$, *change of numéraire*)
- d risky assets S :

$$\frac{dS_t^i}{S_t^i} = \mu_t^i\, dt + \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j\,, \qquad i = 1, \ldots, d$$

μ, σ and σ^{-1} bounded $\mathbb{F}-$adapted with values respectively in $\mathbb{R}^d$ and $\mathcal{S}_{\mathbb{R}}^d$

- *Wealth process* $X^{x,\pi}$, under *self-financing* condition, defined by

$$dX_0^{x,\pi} = x \quad \text{and} \quad dX_t^{x,\pi} = \sum_{i=1}^{d} X_t^{x,\pi} \pi_u^i \frac{dS_u^i}{S_u^i} = X_t^{x,\pi} \pi_u \cdot (\mu_u du + \sigma_u dW_u)$$

- $\pi \in \mathcal{A}$: admissible portfolio if

$$\int_0^T |\sigma_u^T \pi_u|^2 du < \infty$$

1.2. PROBLEM FORMULATION, portfolio constraints

Let K be a closed convex (!) subset of $\mathbb{R}^d$ containing 0

• $K-$ admissible portfolio : $\pi \in \mathcal{A}_K$ if

$$\pi \in \mathcal{A} \quad \text{and} \quad \pi_u \in K \quad \text{Leb} \otimes \mathbb{P} - \text{a.s.}$$

Example 1 *No short-selling* : $K = \left\{x \in \mathbb{R}^d \;:\; x^i \geq 0\right\}$

Example 2 *Incomplete market* : $K = \left\{x \in \mathbb{R}^d \;:\; x^{i_0} = 0\right\}$

Example 3 *No borrowing* : $K = \left\{x \in \mathbb{R}^d \;:\; \sum_i x^i \leq 1\right\}$

Example 4 *Rectangular constraints* : $K = \left\{x \in \mathbb{R}^d \;:\; m^i \leq x^i \leq M^i\right\}$

Example 5 *Finite capitalizations* : change model expressing portfolios in terms of number of shares...

1.3. PROBLEM FORMULATION, super-replication

• Contingent claim G : $\mathcal{F}_T$−measurable random variable, we will mainly consider $G = g(S_T)$ with $g : [0, \infty) \longrightarrow \mathbb{R}_+$ l.s.c.

• Super-replication problem

$$V(0, S_0) \;:=\; \inf\left\{x \;:\; X_T^{x,\pi} \geq G \text{ a.s. for some } \pi \in \mathcal{A}_K\right\}$$

$\Longrightarrow$ Stochastic control problem in non-standard form !

$\Longrightarrow$ Compare with unconstrained case (martingale approach)

$\Longrightarrow$ Connection with backward stochastic differential equations

$\Longrightarrow$ Very difficult to reach any *a-priori* regularity result

$\Longrightarrow$ 1st idea : *reduce to the classical setting,* i.e. standard formulation

2.1. DUAL FORMULATION : dual characterization of the constraints

Support function of K :

$$\delta(y) \;:=\; \sup_{x \in K} \; x \cdot y$$

Effective domain of δ :

$$\tilde{K} \;:=\; \left\{ y \in \mathbb{R}^d \;:\; \delta(y) < \infty \right\}$$

Lemma Let K be a closed convex subset of $\mathbb{R}^n$. Then

$$x \;\in\; K \;\iff\; \delta(y) - x \cdot y \;\geq\; 0 \text{ for all } y \in \tilde{K}$$

2.2. DUAL FORMULATION : dual variables

Let $\mathcal{D} := \left\{ \text{bounded } \mathbb{F} - \text{adapted processes with values in } \tilde{K} \right\}$

$$\left.\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\right|_{\mathcal{F}_T} := \exp\left[\int_0^T \sigma_u^{-1} (\nu_u - \mu_u) \cdot dW_u - \frac{1}{2} \int_0^T \left| \sigma_u^{-1} (\nu_u - \mu_u) \right|^2 du \right]$$

By Girsanov's Theorem, the process

$$W_u^\nu := W_u - \int_0^u \sigma_u^{-1} (\nu_u - \mu_u) \, du \qquad 0 \le u \le T$$

is a Brownian motion under P^ν, and

$$d\left(X_t^{x,\pi} e^{-\int_0^t \delta(\nu_u) du} \right) = X_t^{x,\pi} e^{-\int_0^t \delta(\nu_r) dr} \left[- \left(\delta(\nu_t) - \pi_t \cdot \nu_t \right) dt + \sigma_u dW_u^\nu \right]$$

$$\Longrightarrow \text{The process } \left\{ X_t^{x,\pi} e^{-\int_0^t \delta(\nu_u) du}, \; 0 \le t \le T \right\}$$

is a $\mathbb{P}^\nu$−super-martingale for every $\pi \in \mathcal{A}_K$ and $\nu \in \mathcal{D}$

2.3. DUAL FORMULATION :

reducing to a standard stochastic control problem

Theorem $$V(0,S_0) = \tilde{V}(0,S_0) := \sup_{\nu\in\mathcal{D}} \mathbb{E}^{\mathbb{P}^\nu}\left[G\, e^{-\int_0^T \delta(\nu_u)du}\right]$$

<ElKaroui-Quenez 95, Cvitanić-Karatzas 93, Föllmer-Kramkov 99>

$G = g(S_T)$ and S is a Markov diffusion $\Longrightarrow$ Girsanov's Theorem

$$V(0,S_0) = \tilde{V}(0,S_0) := \sup_{\nu\in\mathcal{D}} \mathbb{E}\left[g\left(S_T^\nu\right)\, e^{-\int_0^T \delta(\nu_u)du}\right]$$

where

$$S_0^\nu = S_0 \quad \text{and} \quad dS_t^\nu = \text{diag}\,[S_t^\nu]\left(\nu_t\, dt + \sigma\left(S_t^\nu\right) dW_t\right)$$

Stochastic control problem in standard form

2.4. DUAL FORMULATION : the HJB equation

From general theory, if V is locally bounded, then

$$-(V_*)_t - \frac{1}{2}\text{Tr}\left[\overline{\sigma\sigma}^T(s)D^2V_*\right] - \text{diag}[s]y \cdot DV_* + \delta(y)V_* \;\geq\; 0 \quad \text{for all} \quad u \in \tilde{K}$$

in the viscosity sense (super-solution property), where $\overline{\sigma}(s) := \text{diag}[s]\sigma(s)$

Since $\tilde{K}$ is a cone, this is equivalent to

$$\min\left\{-(V_*)_t - \frac{1}{2}\text{Tr}\left[\overline{\sigma\sigma}^T(s)D^2V_*\right] \,,\, \inf_{y\in\tilde{K}_1}\left(\delta(y)V_* - \text{diag}[s]y \cdot DV_*\right)\right\} \;\geq\; 0$$

where $\tilde{K}_1 := \left\{y \in \tilde{K} \;:\; |y| = 1\right\}$

We will see later that this is the HJB equation for our problem

FROM NOW ON : MARKOV MODEL

- Risky assets dynamics :

$$\frac{dS_t^i}{S_t^i} = \mu^i(t, S_t)\, dt + \sum_{j=1}^{d} \sigma^{ij}(t, S_t)\, dW_t^j\,, \qquad i = 1, \ldots, d$$

μ and σ Lipschitz, linearly growing, and we will usually forget about the dependence upon t.

- Contingent claim

$$G = g(S_T)$$

for some

$$g \;:\; [0, \infty)^d \longrightarrow \mathbb{R} \text{ l.s.c. and bounded from below}$$

3. GEOMETRIC DYNAMIC PROGRAMMING PRINCIPLE

- *Trivial claim :* Let (t,s), x, $\pi \in \mathcal{A}_K$ be such that $X_T^{x,\pi} \geq g\left(S_T^{t,s}\right)$. Then

$$X_\tau^{x,\pi} \;\geq\; V\left(\tau, S_\tau\right) \;\; \text{for every stopping time} \;\; \tau \in [t,T] \text{ a.s.}$$

In fact, we have the following *geometric dynamic programming principle* (without dual formulation)

Theorem. *For all* $(t,s) \in [0,T) \times \mathbb{R}_+$, *and stopping time* $\tau \in [t,T]$ *a.s.*

$$V(t,s) \;=\; \inf\left\{x \;:\; X_\tau^{x,\pi} \geq V\left(\tau, S_\tau\right) \text{ a.s. for some } \pi \in \mathcal{A}_K\right\}$$

$\Longrightarrow$ Super-solution property

Proposition $-\dfrac{\partial V_*}{\partial t} - \dfrac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^T D^2 V_*\right] \geq 0$ *and* $\dfrac{\mathrm{diag}[s] D V_*}{V_*} \in K$

Sketch of proof (super-solution property)

For simplicity, assume

$$V(t,s) \ := \ \underline{\text{min}}\left\{x \ : \ X_T^{x,\pi} \geq g\left(S_T\right) \ \text{ for some } \pi \in \mathcal{A}_K\right\}$$

Then, starting from initial wealth $\widehat{x} := V(t,s)$:

$$X_T^{\widehat{x},\widehat{\pi}} \ \geq \ g\left(S_T^{t,s}\right) \ \text{ for some } \ \widehat{\pi} \in \mathcal{A}_K$$

$\Longrightarrow$ Geometric dynamic programming

$$X_\tau^{\widehat{x},\widehat{\pi}} \ = \ V(t,s) + \int_t^\tau X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u \left[\mu_u du + \sigma_u dW_u\right] \ \geq \ V\left(\tau, S_\tau^{t,s}\right)$$

Let $0 = (V-\varphi)(t,s) = \min(V-\varphi)$ for some smooth $\varphi > 0 \Longrightarrow V(t,s) = \varphi(t,s)$, $V(\tau, S_\tau^{t,s}) \geq \varphi(\tau, S_\tau^{t,s})$, and by Itô's lemma :

$$0 \ \leq \ -\int_t^\tau \mathcal{L}\varphi\left(u, S_u^{t,s}\right) du + \int_t^\tau \sigma_u \left(X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u - \text{diag}[S_u] D\varphi(u, S_u)\right) dW_u^0$$

Sketch of proof (super-solution property), continued

$$0 \;\leq\; -\int_t^\tau \mathcal{L}\varphi\left(u, S_u^{t,s}\right) du + \int_t^\tau \sigma_u \left(X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u - \mathrm{diag}[S_u] D\varphi(u, S_u)\right) dW_u^0$$

1. Set $\tau_h := (t+h) \wedge \inf\{u > t \;:\; |\ln S_u - \ln s| \geq 1\}$, and take expected values $\Longrightarrow$ $-\mathcal{L}\varphi(t,s) \;\geq\; 0$

2. **Lemma.** (Loc. behavior of stoch. int.) *Let* b *be a predictable* $W-$*integrable process satisfying* $\int_0^t b_s \cdot dW_s \geq -C\,t$, $0 \leq t \leq \tau$, *for some* $C > 0$ *and positive stopping time* τ. *Then* $\liminf_{t \searrow 0} \frac{1}{t} \int_0^t |b_s| ds = 0$ $\mathbb{P}-$*a.s.*

$$\Longrightarrow \frac{\mathrm{diag}[s] D\varphi}{\varphi}(t,s) \in K \Longleftrightarrow \inf_{y \in \tilde{K}_1} \left(\delta(y) - \frac{\mathrm{diag}[s] y \cdot D\varphi}{\varphi}(t,s)\right) \geq 0$$

4.1. CHARACTERIZING THE TERMINAL CONDITION :

implications from the HJB equation

We have of course $V(T,s) = g(s)$, by definition. Let

$$\overline{V}(s) \;:=\; \liminf_{(t',s')\to(T,s)} V(t,s) \qquad [= \; V_*(T,s)]$$

Lemma *We have* $\overline{V} \geq g$ *and* $\dfrac{\mathrm{diag}[s]D\overline{V}}{\overline{V}} \in K$*.*

The latter condition might not be satisfied by g*. Then*

$$\overline{V} \;\neq\; g \qquad \textit{in general}$$

Sketch of proof (implications from HJB)

• $g \geq C$ *and l.s.c.* $\Longrightarrow V \geq g$ *(Fatou's lemma)*

• *For* $t < T$,

$$\delta(y)V(t,s) - y \cdot \mathrm{diag}[s]DV(t,s) \ \geq \ 0 \quad \textit{for every} \quad y \in \tilde{K}$$

or equivalently,

$$\alpha \ \longmapsto \ \ln \overline{V}\,(se^{\alpha y}) - \delta(y)\alpha \qquad \textit{is non-decreasing}$$

$\Longrightarrow$ *send* t *to* T...

4.2. CHARACTERIZING THE TERMINAL CONDITION : face-lifting

Lemma $\overline{V}(s) \geq \hat{g}(s) := \sup_{y \in \tilde{K}} g\left(se^{y}\right) e^{-\delta(y)}$

Proof *For every* $y \in \tilde{K}$ *:* $0 \leq \delta(y)\overline{V}(s) - y \cdot \text{diag}[s]D\overline{V}$

$$\Longrightarrow \quad 0 \;\leq\; \delta(y) - \frac{\partial}{\partial \alpha} \ln \overline{V}\left(se^{\alpha y}\right)$$

integrate between $\alpha = 0$ *and* $\alpha = 1$*, and recall* $\overline{V} \geq g$ *:*

$$\overline{V}(s) \;\geq\; \overline{V}\left(se^{y}\right) e^{-\delta(y)} \;\geq\; g\left(se^{y}\right) e^{-\delta(y)}$$

y *is arbitrary in* $\tilde{K}$...

4.3. CHARACTERIZING THE TERMINAL CONDITION :

properties of the face-lifting operator

- $\widehat{g} \geq g$ (*$\widehat{g}$ majorant of g*)

- $\dfrac{\text{diag}[s]D\widehat{g}}{\widehat{g}} \in K$ (*satisfies the constraints*)

- $\widehat{\widehat{g}} = \widehat{g}$ (*"projection" property*)

- *If h is such that $h \geq g$ and $\dfrac{\text{diag}[s]Dh}{h} \in K$, then $h \geq \widehat{g}$ (minimality)*

$\widehat{g}$ is the smallest majorant of g

which satisfies the constraints

4.4. CHARACTERIZING THE TERMINAL CONDITION :

Examples for $d = 1$, $K = [-\ell, u] \ni 0$

European call option $\quad g(s) = (s-\kappa)^+$

$$\hat{g}(s) \;=\; \begin{cases} (s-\kappa) & \textit{pour } s \geq \frac{\kappa u}{u-1} \\ \frac{\kappa}{u-1}\left(\frac{(u-1)s}{\kappa u}\right)^u & \textit{pour } s \leq \frac{\kappa u}{u-1} \end{cases}$$

European put option $\quad g(s) = (\kappa-s)^+$

$$\hat{g}(s) \;=\; \begin{cases} (\kappa-s) & \textit{pour } s \leq \frac{\kappa\ell}{\ell+1} \\ \frac{s}{\ell+1}\left(\frac{\kappa\ell}{(\ell+1)s}\right)^\ell & \textit{pour } s \leq \frac{\kappa u}{u-1} \end{cases}$$

4.5. EXPLICIT RESULT IN THE BLACK-SCHOLES MODEL

<Broadie-Cvitanić-Soner 98, Review of Financial Studies>

Theorem *For constant* σ*, we have* $V(t,s) = \mathbb{E}^{\mathbb{P}^0}_{t,s}[\widehat{g}(S_T)]$*, and the optimal hedging strategy is the classical Black-Scholes hedging strategy of the face-lifted contingent claim* $\widehat{g}(S_T)$

In the more general local volatility model $\sigma(t,s)$ *:*

Theorem *Assume* $\text{int}(K) \neq \emptyset$*. Under some conditions,* V *is the unique (in some class) continuous viscosity solution of the HJB equation*

$$\min\left\{-V_t - \frac{1}{2}\text{Tr}\left[\overline{\sigma\sigma}^T(s)D^2V\right],\ \inf_{y\in\tilde{K}_1}\left(\delta(y)V - \text{diag}[s]y\cdot DV\right)\right\} = 0$$

Proof of Broadie-Cvitanić-Soner's result

Denote $w(t,s) := \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\widehat{g}\left(S_T\right)\right]$

(i) $w(t,s) - V(t,s) \le \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\overline{V}(S_T) - V_*(t,s)\right] = \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\int_t^T \mathcal{L}V_*(u,S_u)\right] \le 0$

(iia) $\delta(y)w(t,s) - y \cdot \text{diag}[s]Dw(t,s) = \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\delta(y)\widehat{g}(S_T) - y \cdot \text{diag}[S_T]D\widehat{g}\left(S_T\right)\right]$

≥ 0 *for all* $y \in \tilde{K}$

(iib) $\mathcal{L}w = 0 \implies$ *set* $\widehat{\pi}_u := \dfrac{\text{diag}[s]Dw(u,S_u)}{w(u,S_u)}$, *and apply Itô's lemma :*

$$
\begin{aligned}
\widehat{g}(S_T) &= w(T,S_T) \\
&= w(t,s) + \int_t^T \mathcal{L}w(u,S_u)du + \int_t^T w(u,S_u)\widehat{\pi}_u \cdot \text{diag}[S_u]^{-1}dS_u \\
&= w(t,s) + \int_t^T w(u,S_u)\widehat{\pi}_u \cdot \text{diag}[S_u]^{-1}dS_u \;=\; X_T^{w(t,s),\widehat{\pi}}
\end{aligned}
$$

Since $\widehat{g} \ge g$, *this implies that* $w(t,s) \ge V(t,s)$

4.6. PROOF OF SUBSOLUTION PROPERTY IN THE LOCAL VOLATILITY MODEL

Consider the simple case $\text{int}(K) \neq \emptyset$*, and show that*

$$\min\left\{ -V_t^* - \frac{1}{2}\text{Tr}\left[\overline{\sigma\sigma}^T(s)D^2V^*\right],\ \inf_{y\in\tilde{K}_1}\left(\delta(y)V^* - \text{diag}[s]y\cdot DV^*\right)\right\} \ \leq\ 0$$

in the viscosity sense. Let $(t_0, s_0) \in [0,T)\times\mathbb{R}_+^d$, $\varphi \in C^2$ *be such that*

$$0\ =\ (V^* - \varphi)(t_0, s_0)\ =\ \max\text{(strict)}\ (V^* - \varphi)$$

and assume to the contrary that

$$f(t_0, s_0)\ :=\ \left(-\varphi_t - \frac{1}{2}\text{Tr}\left[\overline{\sigma\sigma}^T D^2\varphi\right]\right)(t_0, s_0)\ >\ 0$$

$$\textit{and}\qquad \hat{\pi}(t_0, s_0)\ :=\ \frac{\text{diag}[s_0]D\varphi(t_0, s_0)}{\varphi(t_0, s_0)}\ \in\ \text{int}(K)$$

PROOF OF SUBSOLUTION PROPERTY, continued (2)

Define the open neighborhood of (t_0, s_0) *:*

$$\mathcal{N} \;:=\; \{(t,s) \;:\; |(t, \ln s) - (t_0, \ln s_0)| \le 1,\; f(t,s) \ge 0 \text{ and } \widehat{\pi}(t,s) \in K\}$$

Since (t_0, s_0) *is a point of strict maximum of* $V^* - \varphi$*, we have*

$$\max_{\partial\mathcal{N}} \left(\ln V^* - \ln \varphi\right) \;=:\; -3\,\eta \;<\; 0$$

Choose $(t_1, s_1) \in \mathrm{int}(\mathcal{N})$ *so that*

$$|\ln V(t_1, s_1) - \ln \varphi(t_1, s_1)| \;\le\; \eta$$

Take (t_1, s_1) *as initial data for the process* S*, and define*

$$\tau \;:=\; \inf\{u > t_1 \;:\; (u, S_u) \notin \mathcal{N}\}$$

PROOF OF SUBSOLUTION PROPERTY, continued (3)

Consider the initial capital $\widehat{x} := V(t_1, s_1)e^{-\eta}$, *and compute that*

$$\begin{aligned}
\ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V(\tau, S_\tau) &\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V^*(\tau, S_\tau) \\
&\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + 3\eta \\
&\geq \ln X_\tau^{\varphi(t_1,s_1),\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + \eta
\end{aligned}$$

Next observe that

$$\begin{aligned}
\frac{d\varphi(t, S_t)}{\varphi(t, S_t)} &= \frac{\mathcal{L}\varphi(t, S_t)}{\varphi(t, S_t)}\, dt + \widehat{\pi}(t, S_t) \cdot \mathrm{diag}[S_t]^{-1} dS_t \\
&= \frac{\mathcal{L}\varphi(t, S_t)}{\varphi(t, S_t)}\, dt + \frac{dX_t^{\varphi(t_1,s_1),\widehat{\pi}}}{X_t^{\varphi(t_1,s_1),\widehat{\pi}}}
\end{aligned}$$

PROOF OF SUBSOLUTION PROPERTY, continued (4)

Since $\mathcal{L}\varphi \leq 0$ *for* $t \in [t_1, \tau]$, *and* $X_{t_1}^{\varphi(t_1,s_1),\widehat{\pi}} = \varphi(t_1, s_1)$, *this implies that*

$$X_{\tau}^{\varphi(t_1,s_1),\widehat{\pi}} \;\geq\; V(\tau, S_\tau)$$

Hence, starting from the initial capital $\widehat{x} := V(t_1, s_1)e^{-\eta}$, *we have*

$$\begin{aligned} \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V(\tau, S_\tau) &\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V^*(\tau, S_\tau) \\ &\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + 3\eta \\ &\geq \ln X_\tau^{\varphi(t_1,s_1),\widehat{\pi}} - \ln \varphi(\theta, S_\tau) + \eta \geq \eta \end{aligned}$$

thus contradicting the geometric dynamic programming

5. SUPER-REPLICATION IN STOCHASTIC VOLATILITY MODELS

- *Non-risky asset* $S^0 \equiv 1$ *(discounting / change of numéraire)*

- 1 *risky asset* S *defined by :* $\frac{dS_t}{S_t} = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t^1$

Y *non-tradable state variable :* $dY_t = \eta(Y_t)\,dt + \vartheta_1(Y_t)dW_t^1 + \vartheta_2(Y_t)dW_t^2$

W^1, W^2 *are independent Brownian motions*

η, ϑ_1, ϑ_2 *Lipschitz,* μ, σ *bounded and* $\inf_{s,y} \sigma(s,y)^2 + \vartheta_2(s,y)^2 > 0$

- *Wealth process :* $X_0^{x,\pi} = x$ *and* $dX_t^{x,\pi} = X_t^{x,\pi}\,\pi_t\,\frac{dS_t}{S_t}$

where π *adapted and* $\int_0^T |\pi_t|^2 du < \infty$ *a.s.* $\longrightarrow \mathcal{A}$

• *Let* $G = g(S_T) \geq 0$ *be a contingent claim, for some*

$$g \ : \ \mathbb{R} \longrightarrow \mathbb{R}_+ \ \text{ l.s.c. and } \ g(s) \ \leq \ \text{Const}(1+s)$$

Super-replication problem :

$$V(0,S_0) \ := \ \inf\left\{x \ : \ X_T^{x,\pi} \geq G \text{ a.s. for some } \pi \in \mathcal{A}_K\right\}$$

Observe that this corresponds to :

$$d = 2, \ (S^1, S^2) = (S, Y) \ \text{ and } \ K \ = \ \mathbb{R} \times \{0\}$$

in particular $\text{int}(K) = \emptyset$ *(see proof of subsolution property).*

5.1. IMPLICATIONS OF THE SUPER-SOLUTION PROPERTY

Recall

Proposition

$$-\frac{\partial V_*}{\partial t} - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^T D^2 V_*\right] \;\geq\; 0 \quad \textit{and} \quad \frac{\mathrm{diag}[s] D V_*}{V_*} \in K = \mathbb{R} \times \{0\}$$

$\Longrightarrow V_*(t,s,y) = V_*(t,s)$ *independent of* y

$\Longrightarrow -\frac{\partial V_*}{\partial t} - \frac{1}{2} s^2 \sigma^2(y) \frac{\partial^2 V_*}{\partial s^2} \geq 0$ *for every* $y \in \mathbb{R}$,

5.2. THE BOUNDED VOLATILITY CASE

Let $\sigma : \mathbb{R} \longrightarrow \mathbb{R}_+$ *be such that*

$$\underline{\sigma} := \inf_{y\in\mathbb{R}} \sigma(y) \geq 0, \quad and \quad \overline{\sigma} := \sup_{y\in\mathbb{R}} \sigma(y) < \infty$$

Then, in the viscosity sense

$$-\{V_*\}_t - \frac{1}{2}\left[\overline{\sigma}^2 \{V_*\}_{ss}^+ - \underline{\sigma}^2 \{V_*\}_{ss}^-\right] \geq 0 \quad and \quad V_*(T,s) \geq g(s)$$

Proposition V *is the unique continuous viscosity solution of ("Black-Scholes Barrenblatt PDE")*

$$-V_t - \frac{1}{2}\left[\overline{\sigma}^2 V_{ss}^+ - \underline{\sigma}^2 V_{ss}^-\right] = 0 \quad and \quad V(T,s) = g(s)$$

in the class of functions with linear growth in s.

If $\underline{\sigma} > 0$, *then* V *is* $C^{1,2}\left([0,T),\mathbb{R}_+\right) \cap C^0\left([0,T] \times \mathbb{R}_+\right)$

Sketch of proof ($\underline{\sigma} > 0$) 1. The nonlinear PDE

$$(*) \qquad -v_t - \frac{1}{2}\left[\overline{\sigma}^2 v_{ss}^+ - \underline{\sigma}^2 v_{ss}^-\right] \;=\; 0 \;\; \text{and} \;\; v(T,s) \;=\; g(s)$$

has a unique classical solution with linear growth

V *is a supersolution of* $(*)$, $V(T,.) \geq g$ *and* V *has linear growth*

$$\Longrightarrow \quad V \;\geq\; v \qquad \text{(Maximum principle)}$$

2. By Itô's lemma

$$\begin{aligned} g(S_T) \;&=\; v(T,S_T) \;=\; v(0,S_0) + \int_0^T \mathcal{L}^{Y_t} v(t,S_t)dt + \int_0^T S_t v_s(t,S_t)\frac{dS_t}{S_t} \\ &\leq\; v(0,S_0) + \int_0^T S_t v_s(t,S_t)\frac{dS_t}{S_t} \qquad (-\mathcal{L}^y v(t,s) \geq 0 \text{ for every } y) \end{aligned}$$

i.e. $g(S(T))$ *can be super-hedged starting from initial capital* $v(0,S_0)$

$$\Longrightarrow \quad v \;\geq\; V \qquad \text{(by definition of } V\text{)}$$

Robustness of the Black-Scholes formula

< ElKaroui, Jeanblanc and Shreve ? ?, Finance and Stochastics >

Convex payoff : If g is convex (e.g. Call and Put options), then

$$V(t,s) \;=\; E\left[g\left(\overline{S}_T^{t,s}\right)\right] \;\; \text{where} \;\; d\overline{S}_t \;=\; \overline{S}_t\left[\mu\, dt + \overline{\sigma}\, dW_t\right]$$

(Black-Scholes formula with the largest volatility)

Proof *: $(t,s) \longmapsto E\left[g\left(\overline{S}_T^{t,s}\right)\right]$ is $C^{1,2}$ and convex in s...*

Concave payoff :If g is concave, then

$$V(t,s) \;=\; E\left[g\left(\underline{S}_T^{t,s}\right)\right] \;\; \text{where} \;\; d\underline{S}_t \;=\; \underline{S}_t\left[\mu\, dt + \underline{\sigma}\, dW_t\right]$$

(Black-Scholes formula with the smallest volatility)

5.3. THE UNBOUNDED VOLATILITY CASE

< Cvitanić, Pham and Touzi 1999, J. Appl. Prob. > Let be such that

$$\sigma \ : \ \mathbb{R} \longrightarrow \mathbb{R}_+ \,, \quad \underline{\sigma} \ := \ \inf_{y\in\mathbb{R}} \sigma(y) \ \geq \ 0 \,, \ \textit{and} \ \overline{\sigma} \ := \ \sup_{y\in\mathbb{R}} \sigma(y) \ = \ \infty$$

Then, V_* *is concave in* s *and* $-\{V_*\}_t - \frac{1}{2}\underline{\sigma}^2\{V_*\}_{ss} \ \geq \ 0$

in the viscosity sense

Proposition *Let* $\overline{\sigma} = +\infty$ *and*

either g *is convex or* $\underline{\sigma} \ = \ 0$

Then $V(t,s) = g^{conc}(s) =$ **cost of the cheapest Buy-and-hold super-hedging strategy**

Buy-and-hold strategies

$$\text{Recall} \quad dX_t^{x,\pi} := X_t^{x,\pi}\pi_t \frac{dS_t}{S_t} = \phi_t \, dS_t$$

where $\phi_t = \dfrac{X_t^{x,\pi}\pi_t}{S_t}$ *: number of shares of S in portfolio at time t*

$(\mathcal{A}^{BH})$ – *Buy-and-hold strategy : no dynamic trading*

$$\phi_t = \phi_0 \quad \text{and therefore} \quad X_t^{x,\pi} = \phi_0(S_t - S_0)$$

$$\begin{aligned} V^{BH}(0, S_0) &:= \inf\left\{x : X_T^{x,\pi} \geq g(S_T) \text{ for some } \pi \in \mathcal{A}^{BH}\right\} \\ &= \inf\{x : x + \phi_0(S_T - S_0) \geq g(S_T) \text{ for some } \phi_0 \in \mathbb{R}\} \\ &= \inf\{x : x + \phi_0(z - S_0) \geq g(z),\ z \geq 0,\ \text{for some } \phi_0 \in \mathbb{R}\} \\ &= g^{conc}(S_0) \end{aligned}$$

III. OPTIMAL INVESTMENT

UNDER PROPORTIONAL TRANSACTION COSTS

- *Model formulation*

- *Infinite horizon optimal investment*

- *Hamilton-Jacobi-Bellman equation*

- *Quasi-explicit solution*

< Davis and Norman 1990, Mathematics of Operations Research >

1.1. MODEL FORMULATION : Intuition

- S_t *price of the risky asset at time* t
- θ_t *: amount (JD) invested in* S *at time* t
- $\lambda > 0$ *proportional transaction cost parameter*

$$\begin{array}{lccc} \textit{time} & t & \longrightarrow & t+dt \\ \sharp\textit{shares} & \theta_t/S_t & \longrightarrow & \theta_{t+dt}/S_t \end{array}$$

$\Longrightarrow$ *Transaction cost (paid to the Broker) :* $\lambda \left|\theta_{t+dt} - \theta_t\right|$

$\Longrightarrow$ *Total expenses in transaction costs* $\lambda \int_0^T |d\theta_t|$

NEED TO RESTRICT $\{\theta_t\}$ TO HAVE BOUNDED VARIATION

$\theta_t = L_t - M_t$, L *and* M *nondecreasing processes*

1.2. MODEL FORMULATION : Transfer processes

- *1 non-risky asset* $S^0 \equiv 1$ ($r = 0$, *change of numéraire*)
- 1 *risky assets* S *:*

$$dS_t \;=\; S_t\,(\mu\,dt + \sigma\,dW_t)\ , \qquad \mu, \sigma \in \mathbb{R}$$

- L *- adapted nondecreasing :* **cumulated** *transfers from* S^0 *to* S
- M *- adapted nondecreasing :* **cumulated** *transfers from* S *to* S^0
- C *- adapted non-negative process : rate of consumption*
- **Wealth process**

Bank account (S^0) $\quad dX_t \;=\; -C_t\,dt - (1+\lambda)dL_t + (1-\lambda)dM_t$

Risky asset account (S) $\quad dY_t \;=\; \dfrac{Y_t}{S_t}\,dS_t + dL_t - dM_t$

2.1. INFINITE HORIZON OPTIMAL INVESTMENT PROBLEM

Utility function $U(x) := \dfrac{x^p}{p}$, $p \in (0,1)$

Subjective discount factor $\beta > 0$

Our control problem :

$$V(x,y) := \sup_{(L,M,C)\in\mathcal{A}(x,y)} \mathbb{E}\int_0^\infty e^{-\beta t} U(C_t)\,dt$$

where (x,y) *are the initial holdings on the* S^0 *and* S *accounts,*

$$\nu = (L,M,C) \in \mathcal{A}(x,y) \quad \textit{iff} \quad (X^\nu, Y^\nu) \in \mathbf{A}$$

and $\mathbf{A} := \left\{(\xi,\zeta)\in\mathbb{R}^2 \ : \ \xi+\zeta-\lambda|\zeta| \geq 0\right\}$

2.2. INFINITE HORIZON OPTIMAL INVESTMENT PROBLEM : first observations

Property 1 *If* $(1-p)\beta\sigma^2 > p\mu^2$, *then* $V < \infty$ *on* $\mathbf{A}$

Property 2 V *is concave on* $\mathbf{A}$

Property 3 $V = 0$ *on* $\partial\mathbf{A}$

Property 4 *($p-$homogeneity)* *For* $\lambda > 0$ *and* $(x, y) \in A$, *we have*

$$V(\lambda x, \lambda y) \;=\; \lambda^p V(x, y)$$

$\Longrightarrow$ V *is continuous on* $\mathbf{A}$

$\Longrightarrow$ *Reduction to one dimension*

3. Hamilton-Jacobi-Bellman equation

Theorem *(Dynamic programming)* *For every stopping time* τ*,*

$$V(x,y) \;=\; \sup_{\nu=(L,M,C)\in\mathcal{A}(x,y)} \mathbb{E}\left[\int_0^\tau e^{-\beta t}U(C_t)dt + e^{-\beta\tau}V\left(X_\tau^\nu, Y_\tau^\nu\right)\right]$$

In particular, for $\ell, m, c \geq 0$*,* $L_t = \ell t$ *and* $M_t = mt$ *:*

$$\begin{aligned} 0 \;&\leq\; \mathbb{E}\left[V(x,y) - e^{-\beta\tau}V\left(X_\tau^\nu, Y_\tau^\nu\right) - \int_0^\tau e^{-\beta t}U(c)dt\right] \\ &=\; -\mathbb{E}\int_0^\tau e^{-\beta t}\Big[U(c) - \beta V + V_t + (-c-(1+\lambda)\ell + (1-\lambda)m)V_x \\ &\qquad\qquad + (\mu + \ell - m)V_y + \frac{1}{2}\sigma^2 V_{yy}\Big]\left(X_t^\nu, Y_t^\nu\right)dt \end{aligned}$$

By Itô's lemma, in the viscosity sense

$$\Longrightarrow \quad \beta V - V_t - \mu V_y - \frac{1}{2}\sigma^2 V_{yy} \; - \; [U(c) - cV_x]$$
$$+ \; m[V_y - (1-\lambda)V_x]$$
$$+ \; \ell[(1+\lambda)V_x - V_y] \; \geq \; 0$$

for every ℓ, $m, c \geq 0$

Set $\tilde{U}(z) := \sup_{c \geq 0} \; (U(c) - cz) = \dfrac{z^{-q}}{q}$, *where* $p^{-1} - q^{-1} = 1$

Theorem V *is the unique continuous viscosity solution of*

$$\min\left\{\beta V - V_t - \mu V_y - \frac{1}{2}\sigma^2 V_{yy} - \tilde{U}(V_x)\,,\, (1+\lambda)V_x - V_y\,,\, V_y - (1-\lambda)V_x\right\} \; = \; 0$$

on $\text{int}(\mathbf{A})$ *satisfying* $V = 0$ *on* $\partial\mathbf{A}$ *and* $\sup_{(x,y)\in\mathbf{A}} \dfrac{V(x,y)}{(x+y-\lambda|y|)^p} < \infty$

QUASI-EXPLICIT SOLUTION : decomposition of $\mathbf{A}$

• *3 Regions*

$$\mathbf{NT} := \{(x,y) \in \mathbf{A} \ : \ (1+\lambda)V_x - V_y > 0 \text{ and } V_y - (1-\lambda)V_x > 0\}$$

No Transactions region, *maximizer in the HJB equation* $\ell^* = m^* = 0$

$$\mathbf{B} := \{(x,y) \in \mathbf{A} \ : \ (1+\lambda)V_x - V_y \leq 0\}$$

Buy region, *maximizer in the HJB equation* $\ell^* = \infty$ *and* $m^* = 0$

$$\mathbf{S} := \{(x,y) \in \mathbf{A} \ : \ V_y - (1-\lambda)V_x \leq 0\}$$

Sell region, *maximizer in the HJB equation* $\ell^* = 0$ *and* $m^* = \infty$

BANG-BANG BEHAVIOR

QUASI-EXPLICIT SOLUTION : guessing the shapes of the regions

By the $p-$*homogeneity of* V*, we have* $\nabla V(\delta x, \delta y) = \delta^{p-1} \nabla V_x(x, y) \Longrightarrow$

NT, $\mathbf{B}$ and $\mathbf{S}$ are cones

- *Optimal consumption rate* $C^*(x, y) = V_x^{1/(p-1)}$

- L^* *increases only in* $\mathbf{B}$ *with infinite speed : in the* (x, y) *plane, move immediately to* $\partial \mathbf{B}$ *along the vector* $(-(1+\lambda), 1)$

- M^* *increases only in* $\mathbf{S}$ *with infinite speed : in the* (x, y) *plane, move immediately to* $\partial \mathbf{S}$ *along the vector* $(-1, 1-\lambda)$

QUASI-EXPLICIT SOLUTION : guess the solution in $\mathbf{B}$ and $\mathbf{S}$

- *Reduction to one variable :*

$$\psi(x) \;:=\; V(x,1)\,, \qquad \mathbf{NT}_1 \;:=\; \mathbf{NT} \cap \{(x,1)\} \;=\; (x_S, x_B)$$

$$\mathbf{B}_1 \;:=\; \mathbf{B} \cap \{(x,1)\} \;=\; (x_B, \infty)\,, \qquad \mathbf{S}_1 \;:=\; \mathbf{S} \cap \{(x,1)\} \;=\; (-\infty, x_S) \cap \mathbf{A}$$

- *On* $\mathbf{NT}$ *:* $\beta_3\, x^2 \psi''(x) + \beta_2\, x \psi'(x) + \beta_1\, \psi(x) - \tilde{U}(\psi'(x)) = 0$ *for* $x_R < x < x_S$
- *On* $\mathbf{B}$*, we have* $V(x,y) = V(x - \varepsilon(1+\lambda), y + \varepsilon)$*, and by* $p-$*homogeneity :*

$$\psi(x) \;=\; p^{-1}\, b\; (x + 1 + \lambda)^p \quad \textit{for} \quad x \;>\; x_B$$

- *On* $\mathbf{S}$*, we have* $V(x,y) = V(x + \varepsilon(1-\lambda), y - \varepsilon)$*, and by* $p-$*homogeneity :*

$$\psi(x) \;=\; p^{-1}\, s\; (x + 1 - \lambda)^p \quad \textit{for} \quad x \;<\; x_S$$

QUASI-EXPLICIT SOLUTION : guess the solution in A

$$\psi(x) = \begin{cases} p^{-1}\, b\; (x+1+\lambda)^p \;\; \textit{for } x > x_B \\ \\ p^{-1}\, s\; (x+1+\lambda)^p \;\; \textit{for } x < x_S \end{cases}$$

Solve the second order **ordinary differential equation**

$$\beta_3\, x^2 \psi''(x) + \beta_2\, x\psi'(x) + \beta_1\, \psi(x) - \tilde{U}(\psi'(x)) = 0 \quad \textit{for} \quad x_R < x < x_S$$

$\Longrightarrow$ *family of solutions with* **two free parameters** α, γ

Finally, **determine the constants** $x_B, x_S, b, s, \alpha, \gamma$ *by*

forcing C^0, C^1 **and** C^2 **conditions on** ψ **at the points** x_R **and** x_S

$\Longrightarrow$ *6 equations with 6 unknowns...*

QUASI-EXPLICIT SOLUTION : Verification

Use a verification argument to show that the candidate C^2 *solution*

$$v(x,y) \ := \ y^p \, \psi\left(\frac{x}{y}\right)$$

is indeed the value function of our problem, i.e. $v = V$

Need to exhibit the optimal controls :

- *Optimal consumption rate :* $C^*(x,y) \ := \ V_x(x,y)^{-1/(p-1)}$

- *Optimal transfers :* **local times** *at* $\partial \mathbf{A} = \partial_B \mathbf{A} \cup \partial_S \mathbf{A}$