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Backward Stochastic Differential Equations and Applications in Finance

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1 Standard Backward SDE

1.1 Existence and Uniqueness

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which is defined a d-dimensional Brownian motion $W := (W_t)_{t \leq T}$.

- $(\mathcal{F}_t^W)_{t \leq T}$ the natural filtration of W and $(\mathcal{F}_t)_{t \leq T}$ its completion with the P-null sets of $\mathcal{F}$.

- We define the following spaces :

• $\mathcal{P}_n$ the set of $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued processes on $\Omega \times [0, T]$

- $L_n^2\,(\mathcal{F}_t) = \{\eta\,:\,\mathbb{R}^n - \text{valued r. v. } \ \mathbb{E}[|\eta|^2] < \infty\}$
- $\mathcal{S}_n^2(0,T) = \{\varphi \in \mathcal{P}_n, \varphi \text{ is continuous and } \ \mathbb{E}[\sup_{t\leq T} |\varphi_t|^2] < \infty\}$
- $\mathcal{H}_n^2(0,T) = \{Z \in \mathcal{P}_n \, \text{s.t.}\ \mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty\}$
- $\mathcal{H}_n^1(0,T) = \{Z \in \mathcal{P}_n \, \text{s.t.}\ \mathbb{E}[(\int_0^T |Z_s|^2 ds)^{1/2}] < \infty\}$.

Definition 1.1 *: Let* $\xi_T \in L^2_m(\mathcal{F}_T)$ *be a* $\mathbb{R}^m$*-valued terminal condition and let* g *be a* $\mathbb{R}^m$*-valued coefficient,* $\mathcal{P}_m \otimes \mathcal{B}(\mathbb{R}^m \times \mathbb{R}^{m\times d})$*-measurable. A solution for the m-dimensional BSDE associated with parameters* (g, ξ_T) *is a pair of progressively measurable processes* $(Y, Z) := (Y_t, Z_t)_{t\leq T}$ *with values in* $\mathbb{R}^m \otimes \mathbb{R}^{m\times d}$ *such that :* $Y \in \mathcal{S}^2_m,\ Z \in \mathcal{H}^2_{m\times d}$

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T\,. \tag{1}$$

The differential form of this equation is

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T\,. \tag{2}$$

Hereafter g is called the coefficient and ξ the terminal value of the BSDE.

Under some specific assumptions on the coefficient g, the BSDE (1) has a unique solution. The *standard assumptions* are the following : (**H1**)

(i) $(g(t,0,0))_{t\leq T} \in \mathcal{H}^2_m$

(ii) g is uniformly Lipschitz with respect to (y,z) :

there exists a constant $C \geq 0$ $s.t.$ $\forall (y,y',z,z')$,

$$|g(\omega,t,y,z) - g(\omega,t,y',z')| \leq C(|y-y'| + |z-z'|), \quad a.e.$$

Theorem 1.2 *(Pardoux, Peng [30]) Under the standard assumptions* ***(H1)****, there exists a unique solution* (Y, Z) *of the BSDE with parameters* (g, ξ_T)*.*

Remark 1.3 - $\exists!(Y_t,\ Z_t)_{t \leq T}$ *which is* $\mathcal{F}_t^W$-***adapted*** *and square integrable process.*

- *If* ξ_T *and* $f(t, y, z)$ *are deterministic, then* $Z_t \equiv 0$*, and* (Y_t) *is the solution of ODE*

$$\frac{dY_t}{dt} = -g(t, Y_t, 0), \qquad Y_T = \xi_T\,.$$

If the final condition ξ_T *is random, the previous solution is* $\mathcal{F}_T^W$ *measurable, and so non adapted.*

$\hookrightarrow$ In fact we need to introduce the martingale $\int_0^t Z_s dW_s$ as a control processus to obtain an adapted solution.

Proof :

Step 1 : A Linear case

a) First assume that $g \equiv 0$. A solution of equation(1) satisfies :

$$Y_t = \xi_T - \int_t^T Z_s dW_s, \quad Y_T = \xi_T.$$

So, the process Y is necessarily the L^2-martingale $\mathbb{E}[\xi_T|\mathcal{F}_t]$.

As a direct consequence of the representation theorem w.r.t. a Brownian filtration (see for instance Theorem 4.15 in the book of Karatzas and Shreve (1987)), there exists a unique process $Z \in \mathcal{H}^2_{m\times d}$ such that

$$Y_t = \mathbb{E}[\xi_T|\mathcal{F}_t] = \mathbb{E}[\xi_T] + \int_0^t Z_s dW_s\,.$$

Therefore when $g \equiv 0$, we have proved the existence and uniqueness of a solution for the BSDE (1).

b) The extension to the case where g does not depend on (y, z) :

$$g(t, \omega, y, z) := g(t, \omega) = g(t, \omega, 0, 0) \in \mathcal{H}_m^2$$

is obvious given that $(Y_t + \int_0^t g(s)ds, Z_t)$ is solution of the BSDE with 0-coefficient and terminal value $\int_0^T g(s)ds + \xi_T$.

Exercice : $g(s,\omega,y,z) \equiv g(s,\omega)$ (other method)

$$M_t := E[\xi + \int_0^T g(s,\omega)ds|\mathcal{F}_t], \quad 0 \le t \le T$$

M is a $\in L^2(\mathcal{F}^W)$-martingale

$\Rightarrow \quad \exists\ (z_t)_t \in L^2(\mathcal{F}^W)$-adapted process s.t.

$$M_t = M_0 + \int_0^t z_s\,dW_s \quad \Longrightarrow \quad M_T = M_t + \int_t^T z_s\,dW_s \quad \Longrightarrow$$

$$y_t = \xi + \int_t^T g(s,\omega)ds - \int_t^T Z_s\,dW_s$$

where

$$y_t := E[\xi + \int_t^T g(s,\omega)ds|\mathcal{F}_t]\,.$$

Step 2 : A general Lipschitz coefficient $g(t, \omega, y, z)$

As in the deterministic case, the solution will be obtained as the fixed point of an appropriate application Φ defined on $\mathcal{H}^2_m \times \mathcal{H}^2_{m\times d}$.

We will use an norm involving a weight on time parameterized by a positive constant α. Let $\mathcal{D}_\alpha$ be the space $\mathcal{H}^2_m \times \mathcal{H}^2_{m\times d}$ equipped with the norm :

$$\|(Y, Z)\|_\alpha = \{\mathbb{E}[\int_0^T e^{\alpha s}(|Y_s|^2 + |Z_s|^2)ds]\}^{\frac{1}{2}}.$$

Let Φ be the application from $\mathcal{D}_\alpha$ into $\mathcal{D}_\alpha$ defined by :

$$(u,v) := (u_t, v_t)_{t\leq T} \in \mathcal{D}_\alpha \longrightarrow \Phi(u,v) = (Y_t^{u,v}, Z_t^{u,v})_{t\leq T}$$

where $(Y^{u,v}, Z^{u,v})$ is the $\mathcal{D}_\alpha$-valued solution of the BSDE with coefficient $g^{u,v}(t) = g(t, u_t, v_t)$ (that does not depend on (y,z) and then exists through b)).
As usual, the existence of the fixed point we are interested in lies upon the contraction property of the map Φ, obtained for a norm $\|.\|_\alpha$ with a large enough parameter α.

So let $((u,v),(u',v')) \in \mathcal{D}_\alpha \times \mathcal{D}_\alpha$.

Itô's formula applied to $e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2$ yields to :

$$
\begin{aligned}
&e^{\alpha t}|Y_t^{u,v} - Y_t^{u',v'}|^2 + \int_t^T e^{\alpha s}|Z_s^{u,v} - Z_s^{u',v'}|^2 ds \\
&+ \alpha \int_t^T e^{\alpha s}|Y_s^{u,v} - Y_s^{u',v'}|^2 ds = (M_T - M_t) \\
&+ 2 \int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(g(s, u_s, v_s) - g(s, u'_s, v'_s))ds
\end{aligned}
$$

where

$$
M_t = 2 \int_0^t e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(Z_s^{u,v} - Z_s^{u',v'})dW_s
$$

is a uniformly integrable martingale since using the inequality $2ab \leq a^2 + b^2$

we have

$$\mathbb{E}\Big[\,\{\int_0^T e^{2\alpha s}|Y_s^{u,v}-Y_s^{u',v'}|^2|Z_s^{u,v}-Z_s^{u',v'}|^2 ds\,\}^{\frac{1}{2}}\,\Big]$$

$$\le\; c\,\mathbb{E}\Big[\sup_{0\le s\le T}|Y_s^{u,v}-Y_s^{u',v'}|\,\{\int_0^T|Z_s^{u,v}-Z_s^{u',v'}|^2 ds\,\}^{\frac{1}{2}}\,\Big]$$

$$\le\; \frac{c}{2}\,\mathbb{E}[\sup_{0\le s\le T}|Y_s^{u,v}-Y_s^{u',v'}|^2\,]\;+\;\frac{c}{2}\mathbb{E}\Big[\int_0^T|Z_s^{u,v}-Z_s^{u',v'}|^2 ds\,\Big]\,. \tag{3}$$

But we know that $\sup_{s\le T}|Y_s^{u,v}-Y_s^{u',v'}|$ is square integrable and $(Z^{u,v}-Z^{u',v'})\in\mathcal{H}^2_{m\times d}$, then the process $e^{\alpha s}|(Y_s^{u,v}-Y_s^{u',v'})(Z_s^{u,v}-Z_s^{u',v'})|$ is in $\mathcal{H}^1_d$.

From classical results in martingale theory, the associated stochastic integral is a uniformly integrable martingale, with null expectation.

We now use **(H1)**, i.e. g is Lipschitz continuous with constant C. Indeed, taking the expectation leads to :

$$\mathbb{E}\Big[e^{\alpha t}|Y_t^{u,v} - Y_t^{u',v'}|^2) + \mathbb{E}[\int_t^T e^{\alpha s}|Z_s^{u,v} - Z_s^{u',v'}|^2 ds\Big]$$

$$\leq \mathbb{E}\Big[\int_t^T e^{\alpha s}(-\alpha|Y_s^{u,v} - Y_s^{u',v'}|^2$$

$$+ 2C|Y_s^{u,v} - Y_s^{u',v'}|(|u_s - u'_s| + |v_s - v'_s|))ds\Big]$$

$$\leq \frac{C^2}{\alpha}\mathbb{E}\Big[\int_t^T e^{\alpha s}(|u_s - u'_s| + |v_s - v'_s|)^2)ds\Big]$$

$$\leq \frac{2C^2}{\alpha}\mathbb{E}\Big[\int_t^T e^{\alpha s}(|u_s - u'_s|^2 + |v_s - v'_s|^2))ds\Big]$$

where we have used the polarization formula :

$$-\alpha a^2 + 2Cab = -\alpha(a - \frac{C}{\alpha}b)^2 + \frac{C^2}{\alpha}b^2 \leq \frac{C^2}{\alpha}b^2.$$

Therefore, in terms of the α-norms, using the positivity of both terms of the left hand-side of the above inequality, we obtain :

$$||Z^{u,v} - Z^{u',v'}||_\alpha^2 \leq \frac{2C^2}{\alpha^2}||(u-u',v-v')||_\alpha^2$$

This inequality yields to different estimates on the process $Y^{u,v} - Y^{u',v'}$. The most obvious is that at any time t, $e^{\alpha t}\mathbb{E}\left[|Y_t^{u,v} - Y_t^{u',v'}|^2\right] \leq \frac{2C^2}{\alpha}||(u-u',v-v')||_\alpha^2$. In particular, since $Y_0^{u,v}$ and $Y_0^{u',v'}$ are deterministic, we have :

$$|Y_0^{u,v} - Y_0^{u',v'}|^2 \leq \frac{2C^2}{\alpha}||(u-u',v-v')||_\alpha^2.$$

Moreover, by integrating between 0 and T both hand sides of this inequality, we obtain :

$$||Y^{u,v} - Y^{u',v'}||_\alpha^2 \leq \frac{2C^2T}{\alpha}||(u-u', v-v')||_\alpha^2.$$

In short, there exists $K > 0$ depending only on C and T such : that

$$||(Y^{u,v} - Y^{u',v'}, Z^{u,v} - Z^{u',v'})||_\alpha^2 \leq \frac{K}{\alpha}||(u-u', v-v')||_\alpha^2 \quad (4)$$

Thus for any $\alpha > K$, the map Φ is contracting on the Hilbert space $\mathcal{D}_\alpha$.

The fixed point theorem ensures the existence of a unique pair $(Y, Z) \in \mathcal{D}_\alpha$ such that $\Phi(Y, Z) = (Y, Z)$.

The uniqueness has to be understood $dt \otimes d\mathbb{P}$-a.e. By construction the first component Y^Φ of $\Phi(Y, Z)$ is a continuous process, equal to Y, $dt \otimes d\mathbb{P}$-a.e. and the pair (Y^Φ, Z) is the unique solution of the BSDE (g, ξ_T) whose first component is continuous.□

Picard approximation : (other method for existence of the solution)

- Instead of using a fixed point theorem, we could have used an explicit approximation method, in particular the well-known Picard approximation. The interest of this approach is to construct a sequence converging almost surely to the solution. Actually we have :

Proposition 1.4 *Let us consider the Picard sequence (Y^k, Z^k) recursively defined by $(Y^0 = 0, Z^0 = 0)$, and*

$$-dY_t^{k+1} = g(t, Y_t^k, Z_t^k)dt - Z_t^{k+1}dW_t, \quad Y_T^{k+1} = \xi_T \tag{5}$$

This sequence (Y^k, Z^k) converges to (Y, Z) in $\mathcal{H}_m^2 \otimes \mathcal{H}_{m\times d}^2$, and $d\mathbb{P} \times dt$ almost surely. Moreover, the sequence (Y^k) converges uniformly almost surely.

Proof : Let the Picard sequence (Y^k, Z^k) be recursively defined by $(Y^0 = 0, Z^0 = 0)$,

$$-dY_t^{k+1} = g(t, Y_t^k, Z_t^k)dt - Z_t^{k+1}dW_t, \quad Y_T^{k+1} = \xi_T \tag{6}$$

Then, by the previous estimates (4) applied to $(u_t^k, v_t^k) = (Y_t^{k-1}, Z_t^{k-1})$ and $(Y^k, Z^k) = (Y^{(u^k,v^k)}, Z^{(u^k,v^k)})$, we obtain :

$$\| (Y^{k+1} - Y^k, Z^{k+1} - Z^k) \|_\alpha^2 \leq \frac{K}{\alpha} \| (Y^k - Y^{k-1}, Z^k - Z^{k-1}) \|_\alpha^2 \leq \epsilon^k \| (Y^1, Z^1) \|_\alpha^2 \tag{7}$$

where $\epsilon = \frac{K}{\alpha}$ is a constant < 1.

The norm series are geometric.

Therefore both series $(Y^{k+1} - Y^k)$ and $(Z^{k+1} - Z^k)$ converge in the Hilbert space $\mathcal{D}_\alpha$. In particular, the sequence of partial sums of the series converges in $\mathcal{D}_\alpha$, and $d\mathbb{P} \otimes dt$-$a.e.$.

Moreover, it is possible to obtain the same kind of estimates for the uniform norm of the processes $(Y^k - Y^{k-1})$ defined by $\| Y^k - Y^{k-1} \|_\infty^2 := \mathbb{E}(\sup_{0\leq t\leq T} |Y_t^k - Y_t^{k-1}|^2)$.

Actually observe that for any Itô's semimartingale $\Psi_t = \Psi_0 + \int_0^t \phi_s ds + \int_0^t \sigma_s dW_s$, with ϕ and σ in $\mathcal{H}_m^2$ and $\mathcal{H}_{m\times d}^2$ respectively, we have :

– by the maximal inequality of martingales,

$$\| \int_0^{\cdot} \sigma_s dW_s \|_\infty^2 := \mathbb{E}(\sup_{0\leq t\leq T} | \int_0^t \sigma_s dW_s|^2) \leq 4\mathbb{E}(\int_0^T |\sigma_s|^2 ds) := 4||\sigma||^2_{\mathcal{H}^2_{m\times d}}$$

– the same estimate holds for the semimartingale Ψ since

$$\| \Psi \|_\infty^2 \leq 2\Big(\mathbb{E}(|\Psi_0|^2) + \mathbb{E}((\int_0^T |\phi_s| ds)^2) + \| \int_0^\cdot \sigma_s dW_s \|_\infty^2 \Big)$$

$$\leq 2\mathbb{E}(|\Psi_0|^2) + 2T||\phi||^2_{\mathcal{H}^2_m} + 8||\sigma||^2_{\mathcal{H}^2_{m\times d}}.$$

We then apply this last inequality to the semimartingale $Y^{k+1} - Y^k$ with decomposition $\phi_t = g(t, Y_t^k, Z_t^k) - g(t, Y_t^{k-1}, Z_t^{k-1})$ and $\sigma_t = Z_t^k - Z_t^{k-1}$.

Since g is Lipschitz we obtain :

$$
\begin{aligned}
||\phi||^2_{\mathcal{H}^2} &\leq K||(Y^k - Y^{k-1}, Z^k - Z^{k-1})||^2_{\mathcal{H}^2}, \quad \text{and} \\
\| Y^{k+1} - Y^k \|^2_\infty &\leq 2TK||(Y^k - Y^{k-1}, Z^k - Z^{k-1})||^2_{\mathcal{H}^2} \\
&\quad + 8||Z^k - Z^{k-1}||^2_{\mathcal{H}^2} \\
&\leq K||(Y^k - Y^{k-1}, Z^k - Z^{k-1})||^2_{\mathcal{H}^2} \\
&\leq K\epsilon^{k-1}
\end{aligned}
$$

where K is a universal constant that can change over the inequalities. The last estimate is given by (7) in terms of the α-norm, which is equivalent to the $\mathcal{H}^2$-norm. As a consequence, the series $(Y^{k+1} - Y^k)$ is uniformly convergent.

□

1.2 Linear BSDEs

When the coefficient is linear, we can get explicitly the component Y of the solution.

Proposition 1.5 (Linear BSDE) *Let* (β, μ) *be a bounded* $(\mathbb{R}, \mathbb{R}^d)$*-valued progressively measurable process,* φ *be an element of* $\mathcal{H}_1^2(0,T)$ *and* $\xi_T \in L_1^2(0,T)$*. We consider the following linear BSDE :*

$$-dY_t = (\varphi_t + Y_t\beta_t + Z_t\mu_t)\,dt - Z_t\,dW_t; \qquad Y_T = \xi_T\,. \tag{8}$$

Then :

a) the equation (8) has a unique solution $(Y, Z) \in \mathcal{S}_1^2(0,T) \times \mathcal{H}_d^2(0,T)$, and Y is given explicitly by

$$Y_t = \mathbb{E}\left[\xi_T \Gamma_{t,T} + \int_t^T \Gamma_{t,s}\varphi_s\, ds \,|\, \mathcal{F}_t\right] \tag{9}$$

where $(\Gamma_{t,s})_{s \geq t}$ is the adjoint process defined by the forward linear SDE

$$d\Gamma_{t,s} = \Gamma_{t,s}(\beta_s ds + \mu_s dW_s); \qquad \Gamma_{t,t} = 1 \tag{10}$$

satisfying the flow property

$\forall t \leq s \leq u \quad \Gamma_{t,s}\Gamma_{s,u} = \Gamma_{t,u} \quad P - a.s.$

b) if ξ_T and φ are non-negative, then the process (Y_t) is non-negative. Moreover, if in addition $Y_t = 0$ on $B \in \mathcal{F}_t$, then $a.s.$ on B for any $s \geq t$, $Y_s = 0 \quad \xi_T = 0$ and $\varphi_s = 0,\ P - a.e.$.

Proof : We give just the proof of the first assertion. The proof of the second claim can be found in [13].

By Theorem 1.2 there is a unique solution (Y, Z) of the BSDE (8).

It is easily seen that $\left(\Gamma_t Y_t + \int_0^t \Gamma_s \varphi_s ds\right)_t$ is a local martingale. Moreover, using that $\sup_{s\leq T} |Y_s|$ and $\sup_{s\leq T} |\Gamma_s|$ belong to $L^2(\Omega)$, one can get that $\sup_{s\leq T} |Y_s| \times \sup_{s\leq T} |\Gamma_s|$ belongs to $L^1(\Omega)$. Therefore the local martingale $\left(\Gamma_t Y_t + \int_0^t \Gamma_s \varphi_s ds\right)_t$ is a uniformly integrable martingale, whose t-time value is the

$\mathcal{F}_t$-conditional expectation of its terminal value.

1.3 Markovian BSDE's

Let us now consider the solution of some specific, namely BSDE's driven by a SDE's. Actually we assume that the randomness of the coefficient and the terminal value of that BSDE comes from a diffusion process.

For any given $(t,x) \in [0,T] \times \mathbb{R}^d$, we will denote $(Y_s^{t,x}, Z_s^{t,x},\ 0 \le s \le T)$ the solution of the following BSDE :

$$\begin{cases} -dY_s = g(s, X_s^{t,x}, Y_s, Z_s)1_{[s \ge t]}ds - Z_s dW_s \\ \ Y_T \ = \Psi(X_T^{t,x}) \end{cases} \tag{11}$$

where $(X_s^{t,x},\, 0 \leq s \leq T)$ is a strong solution of a standard Itô stochastic differential equation :

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s \ , \ t \leq s \leq T \\ X_s^{t,x} = x \ , \ 0 \leq s \leq t \, . \end{cases} \tag{12}$$

The system (11) and (12) is called a forward-backward stochastic differential equation (FBSDE). The solution is given by $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x},\, 0 \leq s \leq T)$.

Let us now introduce the following assumptions : **(H2)**

(i) $g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m$

is uniformly Lipschitz in (y, z) with Lipschitz constant C :

$|g(t,x,y_1,z_1) - g(t,x,y_2,z_2)| \leq C[|y_1 - y_2| + |z_1 - z_2|]$.

(ii) b and σ are uniformly Lipschitz with respect to x;

(iii) $x \mapsto (g(t,x,0,0), \Psi(x))$ is continuous

and there exists a constant c s.t.

for any (s,x), $|\sigma(s,x)| + |b(s,x)| \leq c(1 + |x|)$ and

$|g(s,x,0,0)| + |\Psi(x)| \leq c(1 + |x|^p)$ for a real constant $p \geq 1/2$.

We will show that the solution $(Y_s^{t,x},\, Z_s^{t,x})$ is Markovian in the sense that those processes can be expressed via a deterministic function of s and $X_s^{t,x}$. Actually we have :

Theorem 1.6 *Under* $(H2)$ *there exist two measurable deterministic functions* $u(t,x)$ *and* $d(t,x)$ *such that the solution* $(Y_s^{t,x}, Z_s^{t,x})$ *of BSDE (11) is given by :*

$$Y_s^{t,x} = u(s, X_s^{t,x}) \ \ \text{and} \ \ Z_s^{t,x} = d(s, X_s^{t,x})\, \sigma(s, X_s^{t,x}), \ \ ds \otimes d\mathbb{P} - a.e.$$

Furthermore, for any $\mathcal{F}_t$*-measurable random variable* $\chi \in L^2$*, the solution* $Y_s^{t,\chi}$ *and* $Z_s^{t,\chi}$ *are given by* $u(s, X_s^{t,\chi})$ *and* $d(s, X_s^{t,\chi})$*, for* $s \geq t$*.*

Proof : a) Let first assume that g does not depend on (y, z), *i.e.*, $g(t, x, y, z) = g(t, x)$. The result can be obtained from a result by Cinlar et al. [6] (one can see also [10]). More precisely, the claim is given by Theorem 6.27 in [6] which in our case can be written as follows :

Lemma 1.7 *Let $\mathcal{B}_e$ be the filtration on $\mathbb{R}^n$ generated by the functions $\mathbb{E}[\int_t^T \phi(s, X_s^{t,x})ds]$ where ϕ is a continuous bounded $\mathbb{R}^n$-valued function. Then the semimartingale $Y_s^{t,x} = \mathbb{E}[\Psi(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x})dr|\mathcal{F}_s]$ admits a continuous version given by $u(s, X_s^{t,x})$ where $u(t, x) = \mathbb{E}[\Psi(X_T^{t,x}) + \int_t^T g(r, X_r^{t,x})dr]$ is $\mathcal{B}_e$-measurable. Moreover, $u(t, x) + \int_t^s g(r, X_r^{t,x})dr + Y_s^{t,x}$ is an additive* *martingale which has the following representation :*

$$\int_t^s g(r, X_r^{t,x})dr + Y_s^{t,x} = \int_t^s d(r, X_r^{t,x})\sigma(r, X_r^{t,x})dW_r, \;\; t \leq s \leq T,$$

where $d(t,x)$ *is* $\mathcal{B}_e$*-measurable. In particular one can conclude that*

$$Z_s^{t,x} = d(s, X_s^{t,x})\,\sigma(s, X_s^{t,x}), \;\; ds \otimes d\mathbb{P} - a.e.$$

b) Let us now consider a general Lispchitz coefficient $g(s,x,y,z)$. The construction of the solution $(Y_s^{t,x}, Z_s^{t,x})$ is showed by the iterative procedure used in the proof of Proposition 1.4. Let $(Y^{(t,x),n}, Z^{(t,x),n})$ be the sequence defined recursively by

$$\begin{cases} Y^{(t,x),0} &= 0; \qquad Z^{(t,x),0} = 0, \\ -dY_s^{n+1} &= g(s, X_s^{t,x}, Y_s^n, Z_s^n)ds - Z_s^{n+1}dW_s\,; \qquad Y_T^{n+1} = \Psi(X_T^{t,x}) \end{cases}$$

Using the contracting mapping defined in the proof of Theorem 1.2, we know that the sequence $(Y^{(t,x),n}, Z^{(t,x),n})$ converges in $\mathcal{D}_\alpha \times \mathcal{D}_\alpha$ as $n \longrightarrow +\infty$ to $(Y_s^{t,x}, Z_s^{t,x})$, the unique solution of the BSDE. By applying Lemma 1.7, we conclude by a recursive argument that there exist some $\mathcal{B}_e$-measurable functions u_n, d_n such that :

$$\forall s \in [t,T], \quad Y_s^{(t,x),n} = u_n(s, X_s^{t,x}), \quad Z_s^{(t,x),n} = d_n(s, X_s^{t,x})\sigma(s, X_s^{t,x}).$$

Now using estimate (7) given in the proof of Proposition 1.4, we obtain the control of the norm

$$k_n^{(t,x)} = \| Y^{(t,x),n+1} - Y^{(t,x),n} \|_\alpha^2 + \| Z^{(t,x),n+1} - Z^{(t,x),n} \|_\alpha^2 \leq \epsilon^n k_0^{(t,x)}$$

for a positive constant $\epsilon < 1$. Besides by estimate (1.4), we have

$$k_0^{(t,x)} = \| Y^{(t,x),1} - Y^{(t,x),0} \|_\alpha^2 + \| Z^{(t,x),1} - Z^{(t,x),0} \|_\alpha^2 \leq C(1+|x|^2).$$

Next fix $M \in \mathbb{N}^*$ and put $K_M = \{x \in \mathbb{R}^d \,/\, |x| \leq M\}$. Hence, for each $t \in [0,T]$ and $x \in K_M$, we have

$$k_n^{(t,x)} = \| Y^{(t,x),n+1} - Y^{(t,x),n} \|_\alpha^2 + \| Z^{(t,x),n+1} - Z^{(t,x),n} \|_\alpha^2 \leq \epsilon^n C(1+M^2).$$

From this inequality and Borel-Cantelli's Lemma we deduce that $(Y^{(t,x),n}, Z^{(t,x),n})$ converges $ds \otimes d\mathbb{P} - a.e.$ to $(Y^{(t,x)}, Z^{(t,x)})$. Let us denote :

$$u(s,x) := \limsup_{n \to +\infty} u^n(s,x) \quad \text{and} \quad d(s,x) := \limsup_{n \to +\infty} d^n(s,x).$$

First let us notice that the optional processes $(u(s, X_s^{t,x}), d(s, X_s^{t,x}))_{s \leq T}$ are $ds \otimes d\mathbb{P} - a.e.$ equal to $(Y_s^{(t,x)}, Z^{(t,x)})_{s \leq T}$. In the right hand-side of (11) we can replace $(Y_s^{(t,x)}, Z_s^{(t,x)})$ by $(u(s, X_s^{t,x}), d(s, X_s^{t,x}))$, and $Y_s^{(t,x)}$ appears as the solution of a BSDE with given generator $g(s, X_s^{t,x}, u(s, X_s^{t,x}), d(s, X_s^{t,x}))$. But the previous lemma implies that $Y_s^{(t,x)}$ depends only on $(s, X_s^{t,x})$ that is,

$$Y_s^{(t,x)} = \widehat{u}(s, X_s^{t,x})$$

Moreover, $\widehat{u}(s, X_s^{t,x}) = u(s, X_s^{t,x})\ ds \otimes d\mathbb{P} - a.e.$ and so $\widehat{u}$ may be substituted to u in the generator g. In conclusion $(\widehat{u}^{t,x}(s, X_s^{t,x}), d(s, X_s^{t,x}))$ is solution of the BSDE (11). $\square$

1.4 Connection with PDEs

We go back to the BSDE (11) to provide a probabilistic method for studying solutions of the following semilinear PDE's

$$\begin{cases} (\,\partial_t + \mathcal{L}\,)\, u_t + g(t, x, u_t, D_\sigma u_t) = 0, \ \forall\, t \in [0, T] \text{ and } x \in \mathbb{R}^d, \\ u(T, x) \;=\; \Psi(x) \end{cases} \tag{13}$$

where $\mathcal{L}$ is a second order differential operator given by

$$\mathcal{L} := \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{i,j} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x^i}$$

and $D_\sigma u \;:=\; \nabla u\, \sigma$.

Under the assumption [H2], we recall that $(Y_s^{t,x}, Z_s^{t,x})_{\leq s \leq T}$ denotes the unique solution of the BSDE (11).

Let us stress that $Y_t^{t,x}$ is deterministic since $(Y_s^{t,x}, Z_s^{t,x})$ is adapted with respect the σ-algebra $\mathcal{F}_s^t = \sigma\left(W_r - W_t,\ t \leq r \leq s\right)$ and then $Y_t^{t,x}$ is measurable with respect to the trivial tribe.

In order now to give the connection between solutions of semilinear PDE's (13) and BSDE's (11) we first present a regularity result for solutions of BSDE (see Lemma 2.5 and Theorem 2.9 in [33] for the proof).

Theorem 1.8 *In addition to assumption [H2] we assume that the functions* b, σ, g *and* ψ *are* $\mathcal{C}^3$ *with bounded derivatives. Then :*

i) the process $\{Y_s^{t,x};\, (s,t) \in [0,\, T]^2,\, x \in \mathbb{R}^d\}$ *has a version which belongs a.s. to* $C^{0,0,2}([0,\, T]^2 \times \mathbb{R}^d)$. *In particular* $Y_t^{t,x}$ *is of class* C^2 *in* x *a.s and the derivatives up to order 2 are a.s. continuous in* (t, x).

ii) the random field $\{Z_s^{t,x},\, 0 \leq t \leq s \leq T\}$ *has an a.s. continuous version such that*

$$Z_s^{t,x} = \nabla Y_s^{t,x} \left(\nabla X_s^{t,x}\right)^{-1} \sigma(X_s^{t,x})\,. \tag{14}$$

In particular, taking $s = t$, *we obtain* $Z_t^{t,x} = \nabla Y_t^{t,x}\, \sigma(x)$. $\square$

We now give a probabilistic interpretation for solutions of the semilinear PDE (13) using the solution of the Markovian BSDE (11) :

Theorem 1.9 *(Regular coefficients [33]) : Assume that the functions* b, σ, g *and* ψ *are* $\mathcal{C}^3$ *with bounded derivatives.*
Then :
i) if u *belongs to* $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^m)$ *is a classical solution of PDE (13), then* $(u(s,X_s^{t,x}),D_\sigma u(s,X_s^{t,x}))$ *is the solution of the BSDE (11) in the time interval* $[t,T]$*. In addition for any* $t\le T$, $u(t,x)=Y_t^{t,x}$

ii) if $\{(Y_s^{t,x},\ Z_s^{t,x})\}$ *is the unique solution of the BSDE (11), then* $u(t,x):=Y_t^{t,x}$, $0\le t\le T$, $x\in\mathbb{R}^d$ *belongs to* $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^m)$ *and is a classical solution of the PDE (13).*

Proof :

i) This result is a direct consequence of Itô's formula for $u(s, X_s^{t,x})$, $s \in [t, T]$. ii) The second result is the non-linear Feynman-Kac's formula for the semilinear PDE (13). For the complete proof one can see Pardoux & Peng [33] or Peng [32]. Actually for $t \in [0, T]$ and $x \in \mathbb{R}^d$, let $u(t, x) := Y_t^{t,x}$. From Theorem 1.8, the function u belongs to $\mathcal{C}^{0,2}([0, T] \times \mathbb{R}^d)$. Using the flow property of $\{X_s^{t,x};\ t \leq s \leq T,\ x \in \mathbb{R}^d\}$ and the uniqueness of the solution of the BSDE (11), we get for each $t \leq t + h \leq T$,

$$Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}} = u(t + h, X_{t+h}^{t,x}) \,.$$

Furthermore let $\Delta := \{t = t_0 < t_1 < ... < t_n = T\}$ be a subdivision of the interval $[t,\, T]$,

$$
\begin{aligned}
h(x) - u(t,x) = & -\sum_{i=0}^{n-1} \big(u(t_{i+1},x) - u(t_i,x) \big) \\
= & -\sum_{i=0}^{n-1} \Big(u(t_{i+1},x) - u(t_{i+1}, X^{t,x}_{t_{i+1}}) \Big) \\
& - \sum_{i=0}^{n-1} \Big(u(t_{i+1}, X^{t,x}_{t_{i+1}}) - u(t_i,x) \Big)
\end{aligned}
$$

By Itô's formula and the BSDE (11) we get :

$$
\begin{aligned}
h(x) - u(t,x) == & -\sum_{i=0}^{n-1} \Big[\int_{t_i}^{t_{i+1}} \Big(\mathcal{L}u(t_{i+1}, X^{t_i,x}_r) + f(r, X^{t_i,x}_r) \Big)\, dr \\
& - \int_{t_i}^{t_{i+1}} \Big(Z^{t_i,x}_r - (\nabla u \sigma)(t_{i+1}, X^{t_i,x}_r) \Big)\, dW_r \Big]
\end{aligned}
$$

Now let $\lim\limits_{n\to\infty}\sup\limits_{i}(t_{i+1}^{n}-t_{i}^{n})=0$ in the above equation and using Theorem 1.8, we obtain :

$$h(x)-u(t,x)=-\int_{t}^{T}\big(\,\mathcal{L}u(r,x)\;+f(r,x,u(r,x),(\nabla u\sigma)(r,x)\,)\big)\,dr$$

which implies that $u\in C^{1,2}([0,\,T]\times\mathbb{R}^{d}\,)$ and solves the PDE (13).□

For smooth coefficients the PDE (13) has a classical solution but if we assume that the coefficient g is just a Lipschitz continuous function, one has to consider solutions in a weak sense *i.e.* u becomes a weak solution of equation (13) with terminal value ψ, if the following relation holds :

$\forall \phi \in C_c^{1,\infty}([0,T] \times \mathbb{R}^d)$,

$$\int_t^T (u_s, \partial_t \phi) ds + (u(t,\cdot), \phi(t,\cdot)) - (\Psi(\cdot), \phi(\cdot, T)) + \int_t^T \mathcal{E}(u_s, \phi_s) ds$$
$$= \int_t^T (g(s,\cdot,u_s,\nabla u_s \sigma), \phi_s) ds \tag{15}$$

where $(\phi, \psi) = \int_{\mathbb{R}^d} \phi(x)\psi(x)dx$ denotes the scalar product in $L^2(\mathbb{R}^d, dx)$ and

$$\mathcal{E}(\psi, \phi) = \int_{\mathbb{R}^d} ((\sigma^* \nabla \psi)(\sigma^* \nabla \phi) + \phi \nabla ((\frac{1}{2}\sigma^* \nabla \sigma + b)\psi))dx$$

is the energy of the system associated with the PDE.

So assume that instead of the assumptions [H2-ii] and [H2-iii], the following ones hold true :

- [H2-ii'] : the function b (resp. σ) is $\mathcal{C}^2$ (resp. $\mathcal{C}^3$) with bounded derivatives
- [H2-iii'] : $\Psi \in L^2(\mathbb{R}^d)$ and $g(t,x,0,0) \in L^2([0,T]\times\mathbb{R}^d)$.

Under assumptions [H2-i], [H2-ii'] and [H2-iii'], the BSDE (11) still have a unique solution provided that we give a sense to $\Psi(X_T^{t,x})$ and $g(s, X_s^{t,x}, y, z)$. Indeed this difficulty was tackled in [3] and [2] by using the following norm equivalence result :

Proposition 1.10 *(A norm equivalence result) : There exists two constants* $c_1, c_2 > 0$*, such that for every* $t \le s \le T$ *and* $\phi \in L^1(\mathbb{R}^d)$*, we have*

$$c_1 \int_{\mathbb{R}^d} |\phi(x)|\, dx \le \int_{\mathbb{R}^d} E(\big|\phi(X_s^{t,x})\big|)dx \le c_2 \int_{\mathbb{R}^d} |\phi(x)|\, dx\ .$$

Moreover, for every $\psi \in L^1([0,T]\times\mathbb{R}^d)$, $\mathbb{R}$

$$c_1 \int_{\mathbb{R}^d}\int_t^T |\psi(s,x)|dsdx \le \int_{\mathbb{R}^d}\int_t^T E(|\psi(X_s^{t,x})|)dsdx \le c_2 \int_{\mathbb{R}^d}\int_t^T |\psi(s,x)|dsdx\,.$$

The constants c_1, c_2 *depend on* T, *and the bounds of the first (resp. first and second) derivatives of* b *(resp.* σ*).*

The natural space for Sobolev's solutions of semilinear PDE's (15) is the Dirichlet one, namely

$$\mathcal{H} := \{u \in L^2([0,T]\times\mathbb{R}^d)) \,|\, \nabla u\sigma \in L^2([0,T]\times\mathbb{R}^d)\,\}\,.$$

The goal of Proposition 1.10 is to make a natural connection between the norms used to study the BSDE, namely the $\mathcal{H}^2$-norm and the one used in PDE, namely the Sobolev norm.

Barles and Lesigne [3] and Bally and Matoussi [2] proved that the solution of BSDE (11) gives the probabilistic interpretation for the solution of PDE (15)

Theorem 1.11 *(Barles and Lesigne [3], Bally and Matoussi [2]) Assume [M-i], [M-ii'] and [M-iii'] hold. There exists a unique solution $u \in \mathcal{H}$ of the PDE (15). Moreover $u(t,x) = Y_t^{t,x}$ and $D_\sigma u = Z_t^{t,x}$, where $\{(Y_s^{t,x}, Z_s^{t,x}), t \le s \le T\}$ is solution of BSDE (11) and*

$$Y_s^{t,x} = u(s, X_s^{t,x}), \qquad Z_s^{t,x} = D_\sigma u(s, X_s^{t,x}), \quad \forall\, s \in [t, T], \quad dx - a.e.. \tag{16}$$

Proof : The proof is based on the approximation of the PDE (15) by a sequence of PDE's with regular coefficients g_ϵ and regular terminal values Ψ_ϵ for which Theorem 1.9 ii) holds and then to pass to the limit by using the norm equivalence result of Proposition 1.10.

Remark 1.12 *If we assume less regularity on* σ *and* b*, namely uniformly Lipschitz with respect to* x*. Then in the 1-dimensional case (*$m = 1$*) we can prove that* $u(t,x) = Y_t^{t,x}$ *is a viscosity solution of PDE (13). The reader can found a complete and nice study of this point of view in Pardoux [31].*□

1.5 Comparison Theorem

In the one-dimensional case, *i.e.* when $m = 1$, and under the previous assumption **(H1)**, we have a comparison result between the $Y's$ as soon as we can compare the associated coefficients and terminal values.

Theorem 1.13 *(El-Karoui, Peng, Quenez [13]) Let us consider two BSDEs associated with parameters* (g, ξ_T) *and* (g', ξ'_T) *which satisfies* ***(H1)****. We denote by* (Y, Z) *and* (Y', Z') *their respective solutions.*
Assume that $P - a.s.,\ \xi_T \leq \xi'_T$ *and that* $dt \otimes d\mathbb{P}$ *a.e.,* $g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t)$*. Then*

$$Y_t \leq Y'_t, \quad \forall t \in [0, T] \quad P\, a.s.$$

Proof : First observe that the condition $g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t) \leq 0$ and the Lipschitz property of g imply that :

$$g(t, Y_t, Z_t) - g'(t, Y'_t, Z'_t) \leq g(t, Y_t, Z_t) - g(t, Y'_t, Z'_t) \leq C(|Y_t - Y'_t)| + |Z_t - Z'_t|).$$

Applying Itô-Tanaka's formula to $((Y_t - Y_t')^{+,2}$ yields to :

$$e^{\alpha t}((Y_t - Y_t')^+)^2 = e^{\alpha T}(\xi_T - \xi_T')^{+,2} - 2\int_t^T e^{\alpha s}(Y_s - Y_s')^+(Z_s - Z_s')dW_s$$

$$V_{t,T} = \int_t^T e^{\alpha s}\Big(- \alpha(Y_s - Y_s')^{+,2} - \mathbf{1}_{\{Y_s > Y_s'\}}|Z_s - Z_s'|^2 + (Y_s - Y_s')^+[g(s, Y_s,$$

$$V_{t,T} \leq \int_t^T e^{\alpha s}\Big(- \alpha(Y_s - Y_s')^{+,2} - \mathbf{1}_{\{Y_s > Y_s'\}}|Z_s - Z_s'|^2 + 2C(Y_s - Y_s')^+(|Y_s -$$

By the polarization formula, for a large enough α and for any a, b we have :

$$-\alpha a^{+,2} - \mathbf{1}_{\{a>0\}}b^2 + 2Ca^+(|a| + |b|) = -\mathbf{1}_{\{a>0\}}(-\alpha|a|^2 - |b|^2 + 2C|a|(|a| +$$

$$= \mathbf{1}_{\{a>0\}}(-(|b| - C|a|)^2 - (\alpha - 2C -$$

Therefore going back to the above inequality and taking $\alpha > 2C + C^2$ to obtain $V_{t,T} \leq 0$.

By the same argument as in the proof of Theorem 1.2, the martingale $\int_t^T e^{\alpha s}(Y_s - Y_s')^+(Z_s - Z_s')dW_s$ is uniformly integrable, with null expectation. Taking into account the assumption on the terminal conditions, we deduce that $\mathbb{E}[e^{\alpha t}(Y_t - Y_t')^{+,2}] \leq 0$ and therefore the desired inequality. $\square$

1.5.1 Financial Mathematics : Pricing of an European option

Let us consider a market where there are one non risky asset P^0 and one finite family of risky assets P^i $(i = 1, ..., m)$:

$$\begin{cases} dP_t^0 = P_t^0 r_t dt \\ dP_t^i = P_t^i[\mu_t^i dt + \sigma_t^i dW_t], t \leq T\,; P_0 > 0 \text{ and } P_t^i > 0. \end{cases}$$

The processes r and μ^i are supposed $\mathcal{P}_1$-measurable and bounded. Let $\pi = (\pi_t^0, \pi_t^1, \ldots, \pi_t^m)_{t \leq T}$ be a portfolio for a small investor. The process π^0 (resp. π^i) is the amount invested in the bound (resp. risky asset i). The process $V_t = V_t^\pi := \pi_t^0 + \pi_t^1 + ... + \pi_t^m$ is the value of the portfolio π. On the other hand if the portfolio is self-financing then (at least formally) we have :

$$
\begin{aligned}
dV_t &= (\pi_t^0)\frac{dP_t^0}{P_t^0} + (\pi_t^1)\frac{dP_t^1}{P_t^1} + ...(\pi_t^m)\frac{dP_t^m}{P_t^m} \\
&= r_t V_t\, dt + \textstyle\sum_{i=0}^m (\pi_t^i(\mu_t^i - r_t)dt + \pi_t^i \sigma_t^i dW_t) \qquad (17) \\
&= r_t V_t\, dt + \pi_t(\mu_t - r_t)dt + \pi_t \sigma_t dW_t).
\end{aligned}
$$

A self-financing portfolio is called *admissible* if P-$a.s.$, $\int_0^T |\pi_t^0| dt < \infty$, for any $i \geq 1$ $\int_0^T |\pi_s^i \sigma_s^i|^2 ds < \infty$ and $V_t^\pi \geq 0$ for any $t \leq T$. The set of admissible portfolios is denoted by $\mathcal{A}$. Now let us consider ξ a non negative contingent claim (the wealth) at time T, for instance $\xi = (P_T^1 - K)^+$. The point is that we want to seek for the initial endowment X_0 for the investor allowing him/her to get the wealth ξ at T in making an investment in the market. Formally the value X_0 is defined by :

$X_0 = \inf\{x; \exists(V,\pi) \text{ s.t. } \pi \in \mathcal{A} \text{ satisfies } (17), V_T = \xi \text{ and } V_0 = x\}$.

At this level let us emphasize that even the couple $(V, \pi\sigma)$ satisfies (17) and $V_T = (P_T^1 - K)^+$, since there is a lack of integrability of the processes then we cannot claim that $(V, \pi\sigma)$ is a solution of a BSDE, namely the one which looks like (17).

Now when *Arbitrage free* and *Completeness* hypotheses are satisfied, which roughly speaking turns into the fact that the matrix σ_t with row σ_t^i is invertible and its inverse is bounded, the value of the contingent claim $\xi = (P_T^1 - K)^+$ is given by

Theorem 1.14 *(El-Karoui, Peng, Quenez [13]) Suppose m=d and the matrix* $\sigma_t = (\sigma_t^1, \sigma_t^2, ..., \sigma_t^m)$ *invertible, bounded with* *bounded inverse. Suppose* $0 \leq \xi \in L_1^2(\mathcal{F}_T)$*. The value* X_0 *is equal to* Y_0 *where* $(Y_t, \pi_t\sigma_t)_{t\leq T}$ *is the solution of the following linear BSDE :*

$$\begin{cases} dY_t = \{r_tY_t + \pi_t\sigma_t(\sigma_t)^{-1}(\mu_t - r_t)\}dt + \pi_t\sigma_t dW_t \\ Y_T = \xi. \end{cases} \tag{18}$$

Y_t *is the value of the same contingent claim at* t *and* π_t *the vector of amounts invested in the risky assets. Moreover* Y *is the minimum value of admissible portfolios, i.e.,* $P - a.s.$*,* $\forall t \in [0, T]$*,* $Y_t \leq V_t^\pi$ *where* (V^π, π) *is the solution of the equation (17)* $(\pi \in \mathcal{A})$*.*

Proof : By Theorem 1.2 there exists a unique couple of square integrable processes (Y, Z) solution of the BSDE with linear coefficient $g(t,y,z) = r_t\, y + z(\sigma_t)^{-1}(\mu_t - r_t)$, namely

$$dY_t = \big(r_t Y_t + Z_t(\sigma_t)^{-1}(\mu_t - r_t)\big)\, dt + Z_t dW_t,\ t \leq T;\ \ Y_T = \xi\,.$$

The terminal value ξ is non negative then by Proposition 1.5-b) the process Y is non negative, and the solution is the value of self-financing admissible portfolio. Actually for $t \leq T$ let us set $\pi_t = Z_t \sigma_t^{-1}$ then $\int_0^T |\pi_s \sigma_s|^2 ds < \infty$ and $(Y, \pi\sigma)$ satisfies (18). Besides using a change of probability one can write

$$dY_t = r_t Y_t dt + Z_t d\tilde{W}_t,\ t \leq T; \, Y_T = \xi$$

where $\tilde{W}$ is another Brownian motion under the new probability $\tilde{P}$, namely the risk neutral probability measure. Now using Itô's formula for the discounted process $exp\{-\int_0^t r_s ds\} Y_t$ and taking the conditional expectation with respect to $\tilde{P}$ we deduce that $Y_t \geq 0$, for any $t \leq T$. It follows that the portfolio defined by (π^0,π), where $\pi^0_t = Y_t - \sum_{i=1}^m \pi^m_t$, is admissible.

Now let $\tilde{\pi} = (\tilde{\pi}^0, \tilde{\pi})$ be an admissible portfolio such that $V^{\tilde{\pi}}_T = \xi$. It implies that

$d(Y_t - V^{\tilde{\pi}}_t) = \{r_t(Y_t - V^{\tilde{\pi}}_t) + (\pi_t - \tilde{\pi}_t)\sigma_t(\mu_t - r_t)\}dt + (\pi_t - \tilde{\pi}_t)\sigma_t dW_t, t \leq T;\ Y_T - V$

Therefore, using again Itô formula one can get $(exp\{-\int_0^t r_s ds\}(Y_t - V^{\tilde{\pi}}_t))_{t\leq T}$ is a $\tilde{P}$-local martingale. On the other hand there exists a sequence $(\tau_n)_{n\geq 0}$ of stopping times such that :

$$Y_0 - V^{\tilde{\pi}}_0 = \tilde{\mathbb{E}}[exp\{-\int_0^{\tau_n} r_s ds\}(Y_{\tau_n} - V^{\tilde{\pi}}_{\tau_n})].$$

Now using Fatou's Lemma we obtain that $Y_0 - V_0^{\tilde{\pi}} \leq 0$.

Next let $t \leq T$ be fixed and let us define $\tau_t^n := \inf\{s \geq t, \int_t^s |(\pi_u - \tilde{\pi}_u)\sigma_u|^2 du \geq n\}$. In the same way as previously we obtain

$$V_t^{\tilde{\pi}} - Y_t = \tilde{\mathbb{E}}\left[exp\{-\int_t^{\tau_t^n} r_s ds\}(V_{\tau_t^n}^{\tilde{\pi}} - Y_{\tau_t^n})|\mathcal{F}_t\right].$$

Therefore using once more Fatou's Lemma we get $V_t^{\tilde{\pi}} - Y_t \geq 0$, $P - a.s.$. As the processes $V^{\tilde{\pi}}$ and Y are continuous then $P - a.s.$, $\forall t \leq T$, $Y_t \leq V_t^{\tilde{\pi}}$, thus we have obtained the desired result. □

Remark 1.15 *: If instead of assumptions [H1]-(ii) we just require that g satisfies the following monotonicity condition w.r.t. y :*

$$(g(s,y^1,z) - g(s,y^1,z))(y^1 - y^2) \leq K(y^1 - y^2)^2$$

and, moreover, the mapping $z \mapsto g(t,y,z)$ *is uniformly Lipschitz then the BSDE associated with* (g, ξ_T) *has a unique solution. For more details on that claim one can see e.g. (Pardoux and Darling [8] or Pardoux [31]).*□

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