Introduction to discrete time financial models

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1. Financial markets

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2. The one period binomial model

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3. General discrete time models

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4. Application to multinomial models

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1. Financial markets

1.1. Probabilistic framework

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- $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,T}$ a filtration s.t. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.
- $L^0(\mathcal{F}_t)$ the set of $\mathcal{F}_t$-measurable maps.

1.2. Financial assets

- $B = (B_t)_{t \leq T}$: non-risky asset, i.e. $B_t \in L^0(\mathcal{F}_{t-1})$.

Set $R_t = B_{t+1}/B_t$, $r_t = R_t - 1$ and $B_0 = 1$ by convention.

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- For $G \in L^0(\mathcal{F}_T)$, we set $\tilde{G} = G/B_T$.

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- Wealth dynamics : $X_{t+1}^{x,\phi} = \phi_t \cdot S_{t+1} + \left(X_t^{x,\phi} - \phi_t \cdot S_t\right) B_t^{-1} B_{t+1}$

so that $X_{t+1}^{x,\phi} = X_t^{x,\phi} + \phi_t \cdot \Delta S_{t+1} + \left(X_t^{x,\phi} - \phi_t \cdot S_t\right) r_t$

- Observe : $\tilde{X}^{x,\phi}_{t+1} = \phi_t \cdot \tilde{S}_{t+1} + \left(\tilde{X}^{x,\phi}_t - \phi_t \cdot \tilde{S}_t\right)$

so that $\tilde{X}^{x,\phi}_{t+1} = \tilde{X}^{x,\phi}_t + \phi_t \cdot \Delta\tilde{S}_{t+1}$ and $\tilde{X}^{x,\phi}_{t+1} = x + \sum_{i \le t} \phi_i \cdot \Delta\tilde{S}_{i+1}$

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- Observe : $\tilde{X}_t^{x,\phi} = x + \tilde{X}_t^{0,\phi} =: x + \tilde{X}_t^{\phi}$

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- American : receive $G_t \in L^0(\mathcal{F}_t)$ at t against the payment of a premium but can choose t (before T).

1.5. Main questions

- Under which condition

NA : $\nexists\ \phi \in \mathcal{A}$ s.t. $X_T^{\phi} \geq 0$ and $\mathbb{P}\left[X_T^{\phi} > 0\right] > 0$

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- What is the price of a contingent claim G ?

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- $T = 1$, $d = 1$ and write S for S^1
- $\Omega = \{\omega_u, \omega_d\}$, $\mathbb{P}[\omega_u] = p$, $0 < p < 1$
- $S_1(\omega_u) = S_0 u$, $S_1(\omega_d) = S_0 d$, $u > d$, $S_0 > 0$

2.2. Absence of arbitrage opportunity

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- We define $\mathcal{M}(S) := \{\mathbb{Q} \sim P \ : \ \tilde{S}$ is a $\mathbb{Q}$-martingale$\}$
(i.e. $\mathbb{E}^Q[\tilde{S}_{t+1} \mid \mathcal{F}_t] = \tilde{S}_t$ for $t \leq T - 1$)

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- Cor 2.2.3 : NA $\Leftrightarrow \exists \ Q \in \mathcal{M}(S)$. It is unique and given by $Q(\omega_u) = \pi$.

2.3. Contingent claim hedging and pricing

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- Theo 2.3.4 : Under NA, $p(G) = \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$.

- Def 2.3.5 : We say that p is a viable price for G if there is no $\phi \in \mathcal{A}$ s.t. $X_T^{p,\phi} - G \geq 0$ with $\mathbb{P}\left[X_T^{p,\phi} - G > 0\right] > 0$ or $X_T^{-p,\phi} + G \geq 0$ with $\mathbb{P}\left[X_T^{-p,\phi} + G > 0\right] > 0$

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- Prop 2.3.6 : Under NA the only viable price is $p(G)$.

3. General discrete time model

- Simplification : $|\tilde{S}_{t+1} - \tilde{S}_t| > 0$, $\mathbb{P}[\omega] > 0 \ \forall \omega \in \Omega$

3.1. No-arbitrage

- No local abitrage

NA$_t$: $\nexists \ \phi_t \in L^0(\mathcal{F}_t)$ s.t. $\phi_t \Delta \tilde{S}_{t+1} \geq 0$ and $\mathbb{P}\left[\phi_t \Delta \tilde{S}_{t+1} > 0\right] > 0$

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- Theo 3.1.2 : The assertions are equivalent

(i) NA
(ii) NA$_t$ for all $t = 0, \ldots, T-1$
(iii) $\mathcal{M}(S) \neq \emptyset$

- Let Γ be the set of G such that $\exists\ \phi \in \mathcal{A}$ s.t. $X_T^{\phi} \geq G$ ($\Leftrightarrow \tilde{X}_T^{\phi} \geq \tilde{G}$)

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- Prop 3.1.5 : $\mathrm{NA}_t \Rightarrow \exists\ H_{t,t+1} \in L^0(\mathcal{F}_{t+1})$ s.t. $\mathbb{E}\left[H_{t,t+1} \Delta \tilde{S}_{t+1} \mid \mathcal{F}_t\right] = 0$ and $\mathbb{E}\left[H_{t,t+1} \mid \mathcal{F}_t\right] = 1$

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- Exe 3.1.7 : Assume NA and show that Γ is closed.

3.2. Option pricing

- Theo 3.2.1 : If NA holds then $\mathcal{M}(S) \neq \emptyset$ and

$$\Gamma = \{G \in L^0(\mathcal{F}_T) \; : \; \mathbb{E}^{\mathbb{Q}}[\tilde{G}] \leq 0 \; \forall \, Q \in \mathcal{M}(S)\}$$

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- Theo 3.2.4 : If NA holds then a viable price for G must belong to $[-p(-G), p(G)]$

3.3. Completeness

- Theo 3.3.3 : If the market is complete then $|\mathcal{M}(S)| \leq 1$. Conversely, if $|\mathcal{M}(S)| = 1$ then the market is complete.

4. Application to multinomial models

4.1. Cox-Ross-Rubinstein model

- $\mathbb{P}\left[S_{t+1} = uS_t \mid S_t\right] = p$, $\mathbb{P}\left[S_{t+1} = dS_t \mid S_t\right] = 1 - p$, $u > d$, $0 < p < 1$

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- Exe 4.1.1 : (i) Show that NA implies $u > R > d$
(ii) Show that $\mathcal{M}(S) = \{\mathbb{Q}\}$ with $\mathbb{Q}[S_{t+1} = uS_t \mid S_t] = \pi := (R-d)/(u-d)$
(iii) Show that NA $\Leftrightarrow u > R > d \Leftrightarrow \mathcal{M}(S) = \{\mathbb{Q}\}$

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- Exe 4.2.2 : Compute $p(G)$ under NA

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- Problem : compute the super-replication price

$$p(G) = \inf\{x \in I\!R \ : \ \exists \, \phi \in \mathcal{A}, \ X_t^{x,\phi} \geq G_t \, \forall \, t = 0, \ldots, T\}$$

5.1. Supermartingale and Snell envelop

- Theorem 5.1.2 : If Y is a $\mathbb{Q}$-supermartingale then there is a $\mathbb{Q}$-martingale M and a non-decreasing predictable process A such that $Y_t = M_t - A_t$.

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- Prop 5.1.3 : Define Y by $Y_T = \tilde{G}_T$ and $Y_t = \max\left\{\tilde{G}_t\ ,\ \mathbb{E}^{\mathbb{Q}}[Y_{t+1} \mid \mathcal{F}_t]\right\}$. Then, it is the lowest $\mathbb{Q}$-supermartingale above $\tilde{G}$. We say that Y is the Snell envelop of $\tilde{G}$.

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- Prop 5.1.5 : $\tau^* = \inf\{t \in \{0, \ldots, T\} \ : \ Y_t = \tilde{G}_t\} \in \mathcal{T}$ and $(Y_{t\wedge\tau^*})_t$ is a $\mathbb{Q}$-martingale.

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- Investment : ϕ_t units of $S_t \to \phi_t dS_t$

Remaining part : $(X_t^{x,\phi_t} - \phi_t S_t)B_t^{-1}$ units of $B_t \to (X^{x,\phi_t} - \phi_t S_t)B_t^{-1} dB_t$

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6.1. Black and Scholes model

- One non-risky asset : $B_t = e^{rt}$

- One risky asset : $S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}$

- Contingent claim : $G = g(S_T)$, g bounded continuous.

- Investment : ϕ_t units of $S_t \to \phi_t dS_t$
Remaining part : $(X_t^{x,\phi_t} - \phi_t S_t)B_t^{-1}$ units of $B_t \to (X^{x,\phi_t} - \phi_t S_t)B_t^{-1} dB_t$

- Wealth dynamics : $dX_t^{x,\phi} = (X_t^{x,\phi_t} - \phi_t S_t) r dt + \phi_t dS_t$

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- Solution : $p(0, S_0) = \mathbb{E}^{\mathbb{Q}}[e^{-rT} g(S_T)]$ with $\mathbb{Q} \sim \mathbb{P}$ such that $W_t^Q = W_t + \frac{\mu - r}{\sigma} t$ defined a B.M. under $\mathbb{Q}$.

6.1.2. Rigorous derivation

- Use the martingale representation theorem.

6.2. Approximation of BS model by CRR models

- n periods CRR model with $R_n = e^{rT/n}$, $u_n = e^{b_n + \sigma_n}$, $d_n = e^{b_n - \sigma_n}$, $\sigma_n = \sigma\sqrt{T/n}$, $b_n = bT/n$, $b = (r - \sigma^2/2)$.

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- $S_k^n = S_0 e^{kb_n+\sigma_n \sum_{i=1}^k Z_i^n}$, $(Z_i^n)_i$ iid with $\mathbb{P}\left[Z_1^n = \pm 1\right] = \frac{1}{2}$

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- Theo 6.2.2 : If $g(s) = (K - s)^+$ then $\phi_0^n \to p_s(0, S_0)$.